Introduction to Set Theories

Yifan Shen
A01180750

Occidental College
Mathematics Department

April 2016

Abstract

This presentation serves as an introduction to set theories. One important topic included is continuum hypothesis. Continuum hypothesis is the assertion that there does not exist a set which cardinality is strictly less than that of real set and strictly greater than that of integer set. It is proved that this hypothesis is independent of ZFC set theory (Zermelo–Fraenkel set theory with the axiom of choice included). In order to understand this hypothesis, fundamental axioms (include the axiom of choice) and some important results are introduced. Finally, with the proof of such independence, some further discussions, such as the attempt to add more axioms to construct the dependence, are also included.

Topic Outline

(The outline here is basically an illustration of continuum hypothesis.)

For ZFC set theory, there are 9 axioms. With the axiom of choices, we have the most commonly referred set theory. I shall first introduce the axiom of power set, which would generate our hypoethesis. It states for any set X, there is another set Y that contains all of the subsets of X:

$$\forall X \exists Y \forall Z (Z \subseteq X \implies Z \in Y)$$

One can show that $|Y| = 2^{|X|}$.

Topic Outline (cont.)

In fact, one can show that there is a power set of $\mathbb{Z}$ which is isomorphic to $\mathbb{R}$. That means $|\mathbb{R}| = 2^{|\mathbb{Z}|}$. Then the hypothesis says:

$$\nexists Y(|\mathbb{Z}| < |Y| < |\mathbb{R}|)$$

This hypothesis is proved that one cannot prove or disprove it. In other words, this statement has nothing to do with the current set theory.

References

- Herbert B. Enderton. *Elements of Set Theory*. New York: Academic, 1977.
- P. R. Halmos. *Naive Set Theory*. Springer, 1960.