Linear Algebra

16. Decide if each of the statement is TRUE or FALSE. If FALSE, explain why its false or give a counter example that demonstrates why the statement is false. If TRUE, provide a short proof demonstrating why the statement is true. In all cases A is an $n \times n$ matrix of real numbers.

   (a) $\det(-A) = -\det(A)$

   (b) For $A\vec{x} = \vec{b}$ to have a solution, $\vec{b}$ must be in the rowspace of A.

   (c) A is not invertible if and only if 0 is an eigenvalue of A.

   (d) $(A^T)^T = A$

(a) FALSE.

$\det(-A) = (-1)^n \det A$

So statement is only true if n is odd

(b) FALSE $A\vec{x} = \vec{b}$ has a solution only if $\vec{b}$ is in the column space of A, $\vec{b} \in \text{col}(A)$, not the row space

(c) TRUE $\Rightarrow$ A is not invertible $\Rightarrow$ $\det(A) = 0 \Rightarrow$ There exists a zero eigenvalue since $\det(A) = \prod_{i=1}^{n} \lambda_i$

$\Leftarrow$ If 0 ~~ha~~ is an eigenvalue then $\det(A) = 0$ and since the $\det(A) = 0$, A is not invertible

(d) TRUE $A = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]$ $\quad A^T = \begin{bmatrix} \vec{c}_1^{\,T} \\ \vec{c}_2^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix}$ $\quad (A^T)^T = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]$

$\vec{c}_i \in \mathbb{R}^n$ $\qquad = A$

19. Find the orthogonal decomposition of $\mathbf{v}$ with respect to W.

$$\mathbf{v} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad W = \text{span}\left(\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right)$$

HINT: Recall that the projection matrix P onto a subspace spanned by the columns of matrix A is given by $P = A(A^TA)^{-1}A^T$.

$$A = \begin{pmatrix}1&0\\-1&1\\1&1\\0&1\end{pmatrix} \quad A^T = \begin{pmatrix}1&-1&1&0\\0&1&1&1\end{pmatrix} \quad A^TA = \begin{pmatrix}3&0\\0&3\end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{9}\begin{pmatrix}3&0\\0&3\end{pmatrix} = \begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}$$

$$P = \begin{pmatrix}1&0\\-1&1\\1&1\\0&1\end{pmatrix}\begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}1&-1&1&0\\0&1&1&1\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\\1&1\\0&1\end{pmatrix}\begin{pmatrix}1/3&-1/3&1/3&0\\0&1/3&1/3&1/3\end{pmatrix}$$

$$= \begin{pmatrix}1/3&-1/3&1/3&0\\-1/3&2/3&0&1/3\\1/3&0&2/3&1/3\\0&1/3&1/3&1/3\end{pmatrix} \qquad \vec{p} = P\vec{v} = \begin{pmatrix}1/3&-1/3&1/3&0\\-1/3&2/3&0&1/3\\1/3&0&2/3&1/3\\0&1/3&1/3&1/3\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$\vec{p} = \text{proj}_W(\vec{v}) = \begin{pmatrix}1/3\\2/3\\4/3\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\2\\4\\3\end{pmatrix}$$

$$\text{perp}_W(\vec{v}) + \text{proj}_W(\vec{v}) = \vec{v}$$

$$\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v})$$

$$= \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\2\\4\\3\end{pmatrix} = \begin{pmatrix}2/3\\1/3\\-1/3\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}$$

Check orthogonality

$$\frac{1}{3}\begin{pmatrix}1\\2\\4\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}\frac{1}{3} = \frac{1}{9}(2+2-4+0) = 0 \checkmark$$

Check sum

$$\frac{1}{3}\begin{pmatrix}1\\2\\4\\3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}2\\1\\-1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3\\3\\3\\3\end{pmatrix} = \vec{v}$$

18. (a) Write down the definition of the term "linearly independent."

(b) If vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly independent, will $\vec{u}+\vec{v}, \vec{v}+\vec{w}$, and $\vec{u}+\vec{w}$ also be linearly independent? Justify your answer.

(a) A set of vectors $\{\vec{v}_K\}_{K=1}^{n}$ is said to be linearly independent if and only if the only solution to $\sum_{K=1}^{n} c_K \vec{v}_K = \vec{0}$ is the trivial solution $c_1 = c_2 = \dots = c_n = 0$.

(b) $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = 0 \iff c_1 = c_2 = c_3 = 0$

Since $\vec{u}, \vec{v}, \vec{w}$ are linearly independent

$K_1(\vec{u}+\vec{v}) + K_2(\vec{v}+\vec{w}) + K_3(\vec{u}+\vec{w}) = \vec{0}$

$= (K_1+K_3)\vec{u} + (K_1+K_2)\vec{v} + (K_2+K_3)\vec{w} = \vec{0}$

$K_1 + K_3 = 0$
$K_1 + K_2 = 0$
$K_2 + K_3 = 0$

$\Rightarrow K_1 = K_2 = K_3 = 0 \Rightarrow \vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{u}+\vec{w}$ are linearly independent

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

19. Consider

$$A = \begin{bmatrix} 6 & 4 \\ -6 & -4 \end{bmatrix}$$

(a) Confirm that $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors for A.

(b) By diagonalizing A, confirm that an expression for $A^n = \begin{bmatrix} 3(2^n) & 2^{n+1} \\ -3(2^n) & -2^{n+1} \end{bmatrix}$.

$$\begin{pmatrix} 6 & 4 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A = PDP^{-1} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \frac{1}{1}\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 4 \\ -6 & -4 \end{pmatrix} \checkmark$$

$$A^n = PD^nP^{-1} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1\cdot 2^n \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 2^n & 2\cdot 2^n \\ -3\cdot 2^n & -2\cdot 2^n \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 2^n & 2^{n+1} \\ -3\cdot 2^n & -2^{n+1} \end{pmatrix}$$

20. Given

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

(a) Find the rank of A

(b) Find bases for col(A), row(A) and null(A).

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\text{rank}(A) = 3$

$m = 3$

$n = 4$

$\text{rank} + \text{nullity} = n$

$3 + \text{nullity} = 4$

$\text{nullity} = 4 - 3 = 1$

$\dim(\text{null}(A)) = \text{nullity} = 1$

$x + z = 0$

$y - z = 0$

$a = 0$

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \\ a \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \\ 0 \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\text{row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$\text{null}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$