Discrete Mathematics

21. Let R be a relation on the set $\mathbb{R}^2$ defined by

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if } x_1 - y_1 = x_2 - y_2.$$

(a) Prove that R is an equivalence relation on $\mathbb{R}^2$.

(b) Describe the equivalence class of the element $(3, 2)$ both in set-builder notation and geometrically.

(c) Describe geometrically how the equivalence classes of R partition the plane $\mathbb{R}^2$.

(a) Reflexive

$(x,y)R(x,y) \Leftrightarrow x-y=x-y$

Symmetric

$(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1-y_1=x_2-y_2 \Rightarrow x_2-y_2=x_1-y_1$
$\Leftrightarrow (x_2,y_2)R(x_1,y_1)$

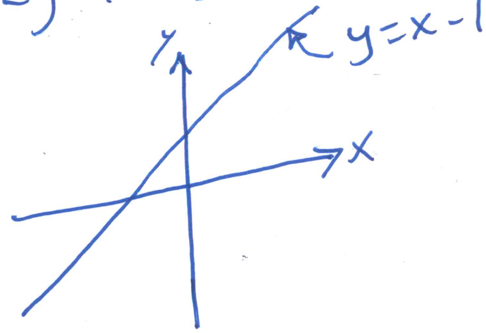
Transitive

$(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1-y_1=x_2-y_2 \Rightarrow x_1-y_1=x_3-y_3$
$(x_2,y_2)R(x_3,y_3) \Leftrightarrow x_2-y_2=x_3-y_3 \quad \Leftrightarrow (x_1,y_1)R(x_3,y_3)$

(b) The equivalence class of $(3,2)$

$[(3,2)]R(x,y) \Leftrightarrow 3-2=x-y \Rightarrow x-y=1$

Geometrically $y=x-1$ is a line in $\mathbb{R}^2$



The class is

$\{(x,y)\in\mathbb{R}^2 \mid y=x-1\}$

(c) The set of all equivalence classes

$(a,b)R(x,y) \Leftrightarrow a-b=x-y \Rightarrow y=x-(a-b)$
$=x-a+b$

are represented as the lines of slope one in $\mathbb{R}^2$

22. Prove the following is a tautology for statements p, q and r:

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

p	q	r	$p \to q$	$q \to r$	$p \to q \wedge q \to r$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

23. Consider the letters (or characters) from the standard 26-member English alphabet.

(a) What is the number of character strings containing six letters?

(b) What is the number of character strings containing six letters if no letter gets repeated in each string?

(c) What is the number of six-letter strings with exactly two vowels (where the set of vowels in English is defined to be the 7 letters $\{a, e, i, o, u, w, y\}$)?

(d) What is the number of six-letter strings with the letter a?

(HINT: You do not need to numerically simplify your calculations.)

(a) 26 ways to choose each place in the ~~string~~

$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$

(b) If letters don't get repeated, number of choices decreases

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = \frac{26!}{20!} = {}_{26}P_{20}$

(c) We have to choose

2 vowels out of 7 possibilities

4 consonants out of $26-7=19$ possibilities

$\binom{6}{2} \cdot 7^2 \cdot 19^4$

If instead of letters (vowels & consonants) we were picking 2 ~~people~~ men out of 7 males & 6 women out of 19 females

answer would be $\binom{6}{2}\binom{19}{7} = \frac{6!}{4!2!} \cdot \frac{19!}{12!7!}$

(d) Number with one a is

Total number of possible choices minus total ~~number~~ number without an a

$26^6 - 25^6$

24. Let A be a set of n elements. The power set of A, $P(A)$ is the set of all subsets of A.

(a) How many elements are there in $P(A)$? Prove it.

(b) Prove or disprove: $P(A \cup B) = P(A) \cup P(B)$.

(a) Power Set of A has $2^{|A|}$ elements where $|A|$ is number of elements in A.

Proof by Induction

① Base Step

$T(1)$ is true

$A = \{1\}$ $\quad P(A) = \{\{1\}, \{\}\}$ $\quad |A| = 1$ $\quad |P(A)| = 2 = 2^{|A|} = 2^1$

A

② Inductive Step

$T(n) \Rightarrow T(n+1)$

Suppose A has n elements, we assume $|P(A)| = 2^{|A|} = 2^n$

If B has n+1 elements $\{1, \ldots, n, n+1\}$ then $P(B)$ looks like $P(\{1, \ldots, n\})$ and $P(n+1)$

You have a set with n elements, which you either add n+1th or not, which produce $2 \cdot 2^n$ subsets $= 2^{n+1}$

(b) $P(A \cup B) \neq P(A) \cup P(B)$

Let $A = \{1\}$

$B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$

$P(A \cup B) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{\}\}$

$P(A) = \{\{1\}, \{\}\}$ $\quad |P(A)| = 2$

$P(B) = \{\{2\}, \{3\}, \{2,3\}, \{\}\}$

$|P(B)| = 4$

$|P(A) \cup P(B)| = 6 \neq 8$

$|P(A \cup B)| = 8 = 2^{|A \cup B|} = 2^3$

25. Prove by induction for all integers $n \geq 2$:

$$\sum_{k=2}^{n} k^2 - k \qquad 2 + 6 + 12 + \cdot + (n^2 - n) = \frac{n(n^2-1)}{3}.$$

$$P(n):\ 2+6+12+n^2-n = \frac{n(n^2-1)}{3}$$

Induction: 1^{st} Prove Base step

$P(2)$: $2^2-2 = 4-2 = LHS$

$RHS = 2 \cdot \frac{(2^2-1)}{3} = 2 \cdot \frac{3}{3} = 2$ ✓

$LHS = RHS$

Inductive Step

$$P(n) \Longrightarrow P(n+1)$$

$$P(n+1) = \underbrace{2+6+12+\ldots+n^2-n}+(n+1)^2-(n+1) = LHS$$

$$= \frac{n(n^2-1)}{3} + n^2+2n+1-n-1 =$$

$$\frac{n(n^2-1)}{3} + (n+1)(n+1) - n+4$$

$$\frac{n(n^2-1)}{3}$$

$$= \frac{(n+1)}{3}[n(n-1)\cancel{+3}(n+1)-3]$$

$$= \frac{(n+1)}{3}[n^2-n+3n+3-3]$$

$$= \frac{(n+1)}{3}[n^2+2n]$$

$$= \frac{(n+1)}{3}[(n+1)^2-1]$$

$= RHS$ ✓