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Solutions to Linear Algebra Section

3-5-07

2005

16. Q: Give an example of a pair of subspaces V and W of $\mathbb{R}^3$ such that $V \neq \{0\}$, $V \neq W$, $V \subset W$, and $W \neq \mathbb{R}^3$

A: $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $W = \begin{bmatrix} x+1 \\ y+2 \\ z+3 \end{bmatrix}$ where $x, y, z \in \mathbb{R}$

17. a) Q: Give the definition of what it means for vector $\vec{v}$ to be an eigen vector of a matrix A

A: a vector $\vec{v}$ is an eigenvector of matrix A if when $\vec{v}$ is multiplied by A, $\vec{v}$ remains of same or becomes a scalar multiple of the original $\vec{v}$

b) Q: Prove or disprove the following statement: If $\vec{v}$ is an eigenvector of a matrix A, then $2\vec{v}$ is also an eigenvector of A

A: $2\vec{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \\ 2v_4 \\ \vdots \\ 2v_n \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ a_{31} & & \ddots & \\ a_{41} & & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$

$$A\vec{v} = c \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_n \end{bmatrix}$$

$$A2\vec{v} = 2c \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_n \end{bmatrix}$$

★ Since when $A \cdot 2\vec{v}$ is $2c$ (where c is a scalar) $\vec{v}$, which is a scalar multiple of $\vec{v}$, that means by definition $2\vec{v}$ is an eigenvector of A

18.Q: Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^3$, and let P be a plane in $\mathbb{R}^3$ containing the origin, such that $\vec{v} \notin P$ and $\vec{v}$ is not orthogonal to P. Let $\vec{w} = \text{proj}_P(\vec{v})$. Answer the following:

A: a) The vector $\vec{w}$ is (i) in P

b) The vector $\vec{v}-\vec{w}$ is (ii) perpendicular to P

c) If $\vec{v} = [1,2,3]$ and P is the xz-plane, what is w?

$$\vec{w} = [1,0,3]$$

19.Q: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Let $B = \text{rref}(A)$

a) Find B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Do A and B have the same nullspace?

A and B do have the same nullspace, it is $\vec{0}$

c) Do the systems $A\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have the same solutions?

No because $\vec{x}_1 = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{x}_2 = B^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

but $A^{-1} \neq B^{-1}$ (do not equal eachother) $\therefore X_1 \neq X_2$

20. a) Q: Prove or disprove: For any pair of $n \times n$ matricies A and B, $\det(A+B) = \det(A) + \det(B)$

A: Counter-example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$ $\det(A) = 1$ $\det(B) = -2$

$\det(A) + \det(B) = -1$

$\det(A+B) = 4$

$\therefore \det(A+B) \neq \det(A) + \det(B)$

b) Q: Prove or Disprove: If A and B are $n \times n$ matricies and $\det(A) = 0$, then AB is not invertible.

A: Since A and B are both $n \times n$ matricies and $\det(A) = 0$, then $\det(AB) = 0$ because a property of determinants is $\det(AB) = \det(A)\det(B)$, and by the Fundemental Theorum of Invertible Matricies if an $n \times n$ matrix has $\det = 0$ then it is not invertible $\therefore$ AB is not invertible

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16. Fill in blank by choosing (a), (b), or (c) and justify why your answer makes sense.
The equation $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is in the <u>(b)</u> of A.
(a) Row space (b) column space (c) null space
We know that $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the column space of A because $A\vec{x}$ is the linear combinations of the colums ($c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$).

$$\underset{m \times n}{m\overset{n}{\begin{bmatrix} V & & \\ & & \end{bmatrix}}} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Furthermore, we know it cannot be null space because that is the solutions to $A\vec{x} = 0$

17. Let S be the supspace of $\mathbb{R}^3$ spanned by [1,0,0] and [1,0,1]. Describe $S^\perp$ (geometrically & dimension) and give a basis for it

$S = \{[1,0,0], [1,0,1]\}$

The orthogonal complement of the rowspace of A, an mxn matrix, is the nullspace of A, and the orthogonal complement of the column space of A is the nüllspace of A^T.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

x_2 free

(now in column vectors)

$\text{null}(A^T) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

So the basis for $S^\perp = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ so $S^\perp$ is in $\mathbb{R}^3$ and a vector