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Solutions to the Mathematics Comprehensive Examination, Fall 2004

Calculus 2

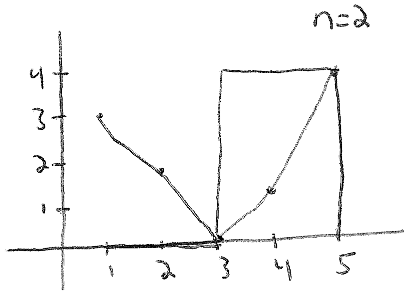
1) We know that $f'(t) = \sin(t^2)$ and $f(0) = 1$. We want to estimate $f(2)$. Of the give choices, we can estimate it using Euler's Method

a. Euler's Method

We have information about $f(0)$ and we want to know $f(2)$. Our formula for Euler's Method is $f(x_{n+1}) \approx f(x_n) + \Delta t * f'(x_n)$. Thus, we can say that $x_n = 0$ and $x_{n+1} = 2$. Then, our $\Delta t = 2$ and $f'(x_n) = \sin(0) = 0$, so $f(x_{n+1}) \approx f(x_n) + 0$, and $f(2) \approx f(0) = 1$, $f(2) \approx \boxed{1.}$

2a) We need to estimate $f'(2)$, or the value of the slope of the tangent line at $x = 2$. We have points for $x = 1, 2$, and 3 and their corresponding y values. So to estimate the slope at $x = 2$, we need to take the difference of the y values at 3 and 1 and divide it by those x-values. This gives us $(3-0)/(1-3) = \boxed{-2/3.}$

2b) n=2



First rectangle has area zero, because $f(3)=0$, and $f(3)$ determines the height of the first rectangle. The second rectangle has area $2 \cdot 4 = 8$, so the approximation for $\int_1^5 f(x)dx$ is $\boxed{8.}$

Interval is from $5-1=4$ units. If $n=2$, ~~that~~ we're using right endpoints $f(3)$ determines the height of the first rectangle and $f(5)$ of the second.

4a)

$$\int_{1/4}^{1/2} f(z)\,dz = \int_{1/4}^{1/2}\Big[\sum_{k=0}^{\infty} z^k\Big]dz = \int_{1/4}^{1/2}\big[1+z+z^2+z^3+\dots\big]dz = z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\dots\Big|_{1/4}^{1/2}$$

$$= \Big[(1/2)+\frac{(1/2)^2}{2}+\frac{(1/2)^3}{3}+\frac{(1/2)^4}{4}+\dots\Big]-\Big[1/4+\frac{(1/4)^2}{2}+\frac{(1/4)^3}{3}+\frac{(1/4)^4}{4}+\dots\Big]$$

$$= \big[1/2 - 1/4\big] + \frac{1}{2}\big[(1/2)^2-(1/4)^2\big] + \frac{1}{3}\big[(1/2)^3-(1/4)^3\big] + \frac{1}{4}\big[(1/2)^4-(1/4)^4\big] + \dots$$

$$= \sum_{k=1}^{\infty}\frac{1}{k}\big[(1/2)^k-(1/4)^k\big] = \sum_{k=0}^{\infty}\frac{1}{k+1}\Big[\Big(\frac{1}{2}\Big)^{k+1}-\Big(\frac{1}{4}\Big)^{k+1}\Big] \quad \boxed{\text{TRUE}}$$

4b) From the last problem, we saw that

$$\int_{1/4}^{1/2} f(z) = \sum_{k=1}^{\infty} \frac{1}{k}\left[\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right].$$ For all $k>0$, it's true that $\frac{1}{k}>0$ and $\left[\left(\frac{1}{2}\right)^k-\left(\frac{1}{4}\right)^k\right]>0$. Thus, this sum cannot possibly be negative, so it cannot equal $-\ln(2)$, which is <0. $\boxed{\text{FALSE}}$

5) $\int x \cdot e^x dx = \int u\, dv$

Integration by parts $\Rightarrow \int u\,dv = u \cdot v - \int v\,du$

choose $u=x$ and $dv=e^x dx$, so $du=dx$ and $v=e^x$.

$$\int u\,dv = xe^x - \int e^x dx = xe^x - e^x + C = \boxed{e^x[x-1]+C}$$

8) a) We do not know when the function is negative or positive, thus we can't conclude $|f(x)|$ and we don't know anything about $\int_{-2}^{4} |f(x)| dx$.

b) $\int_{-2}^{4} f(x)dx = F(x) \Big|_{-2}^{4} = F(4) - F(-2) = -[F(-2) - F(4)] = -[F(x) \Big|_{4}^{-2}] = -[\int_{4}^{-2} f(x)dx]$

Notice that above, $F'(x) = f(x)$. By the above proof, we have that $\int_{4}^{-2} f(x)dx = -\int_{-2}^{4} f(x)dx$. I took the first and last entry and multiplied them both by -1.

then, $\int_{4}^{-2} f(x)dx = -\int_{-2}^{4} f(x)dx = -10$

c) $\int_{-2}^{4} [f(x)+2]dx = F(x) + 2x \Big|_{-2}^{4} [F(4) + 2(4)] - [F(-2) + 2(-2)]$

$= F(4) + 8 - F(-2) + 4 = [F(4) - F(-2)] + 12 = [10] + 12 = \boxed{22}$

We know that $\int_{-2}^{4} f(x)dx = F(4) - F(-2) = 10$, and once again we're saying that $F'(x) = f(x)$

d) We cannot possibly determine this value

a) First, let's find the area

$$\int_1^4 x^{-2}\,dx = -x^{-1}\Big|_1^4 = -\left[\tfrac{1}{4}-1\right] = -\left[-3/4\right] = 3/4.$$

Half of this area is $\frac{3/4}{2} = 3/8$.

Thus, we're looking for an a such that

$$\int_1^a x^{-2}\,dx = 3/8 \quad \text{or} \quad \int_a^4 x^{-2}\,dx = 3/8.$$

$$\int_1^a x^{-2}\,dx = -x^{-1}\Big|_1^a = -\frac{1}{a}+1,$$ this should equal $3/8$.

$$-\frac{1}{a}+1 = 3/8$$

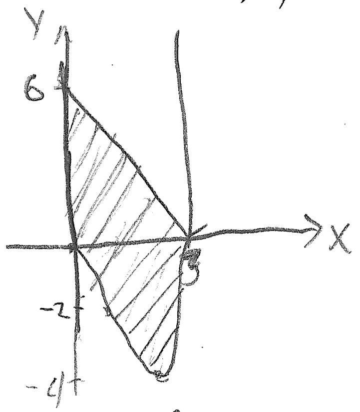
$$-\frac{1}{a} = -5/8$$

$$\frac{1}{a} = 5/8 \Rightarrow \boxed{a = 8/5}$$

Comps, Spring '05, Calc 2

⑥ Express the area of the region enclosed by:

$y = x^3 - 3x^2$, $y = 6 - 2x$, $x = 0$, $x = 3$



$$\text{Area} = \int_0^3 6 - 2x\,dx + \left(-\int_0^3 x^3 - 3x^2\,dx\right) = \int_0^3 -x^3 + 3x^2 - 2x + 6\,dx$$

7. Consider the p-series $\sum_{k=1} \frac{1}{k^3}$ From Spring '05 Calc

(a) Can the Ratio Test be used to determine that this series converges? Explain what happens.

ratio test: $\lim_{k\to\infty} \frac{u_{k+1}}{u_k}$

$$u_{k+1} = \frac{1}{(k+1)^3}, \quad u_k = \frac{1}{k^3}$$

$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^3 = 1$. When the output is 1 it is inconclusive b/c the terms don't decrease fast enough.

(b) Use Integral Test to show series converges

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{r\to\infty} \int_1^r \frac{1}{x^3}\,dx = \lim_{r\to\infty} -\frac{x^{-2}}{2}\Big|_1^r$$

$= \lim_{r\to\infty} -\frac{1}{2x^2} + \frac{1}{2} = \frac{1}{2}$. Since $\frac{1}{2} < 1$, this series converges

⑧ $\int_1^2 \frac{g(u)}{u}\,du = 3$ ① $\int_1^2 g(u)\,du = 4$ ② $\int_1^4 g(u)\,du = 5$ ③ $g(1) = 2$, ④ $g(2) = -2$ ⑤

$\boxed{\text{From Spring '05}}$ Calc

(a) Evaluate

$$\int_1^2 \ln(x)\, g'(x)\, dx$$

Integrate by parts

$u = \ln(x) \rightarrow du = \frac{dx}{x}$

$dv = g'(x)dx \rightarrow v = g(x)$

$$\int_1^2 \ln(x) g'(x) dx = \ln(x) g(x)\Big|_1^2 - \int_1^2 \frac{g(x)}{x} dx = -2\ln(2) - 3$$ by ① and ⑤

(b) Evaluate

$$\int_1^2 x g(x^2) dx$$

u-substitution

$u = x^2$

$du = 2x\,dx \rightarrow dx = \frac{du}{2x}$, "$u(1) = 1$", "$u(2) = 4$"

$$\int_1^2 x g(x^2) dx = \frac{1}{2}\int_1^4 g(u) du = 5/2$$ by ③

(9) From Spring '05 Calc.

x	0	.5	1	1.5	2	2.5	3
$f(x)$	4	7	2	0	2.5	4.5	5

(a) Which integration technique would you expect to produce the most accurate approximation of the area $\int_0^3 f(x)dx$?

Simpson's Rule because it is the average of the left and right limits so clearly is the most accurate

(b) Use right Riemann sum approx. with $n=6$ to estimate $\int_0^3 f(x)dx$

$(7)(0.5)+(2)(0.5)+0+(2.5)(0.5)+(4.5)(0.5)+(5)(0.5)$

$= \frac{21}{2}$

⑩ Evaluate

From Spring '05 Calc

$$\int_0^{\pi} \frac{\sin(x)}{2+\cos(x)}\,dx$$

u-sub

$u = \cos(x)$

$du = \sin(x)\,\cancel{dx}$

$$\int_0^{\pi} \frac{\sin(x)\,dx}{2+\cos(x)} = -\int_1^{-1} \frac{du}{2+u} = \int_{-1}^{1} \frac{du}{2+u} = \ln(2+u)\Big|_{-1}^{1} = \ln(3)$$