Report on Test 2

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Point Distribution (N=21)

Range	100+	90+	85+	80+	75+	70+	65+	60+	55+	50+	45+	40+	40-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	2	4	2	1	1	1	1	2	1	4	0	0	1

Summary Overall class performance was somewhat mediocre. The mean score was 72. The median score was 71. The high score was 105.

#1 Invertible Matrices. These are TRUE or FALSE questions. **(a)**

FALSE The equation $A\vec{x} = \vec{0}$ has a nonzero solution. **If A is invertible $A\vec{x} = \vec{b}$ always has only one solution, $\vec{x} = A^{-1}\vec{b}$.**

TRUE For every vector $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ always has exactly one solution. **See above.**

TRUE The rows of A form a basis for $\mathbb{R}^n$. **The dimension of the row space is the rank of A, which equals n for an invertible matrix. Thus the n rows of A are linearly independent and form a basis for $\mathbb{R}^n$.**

FALSE The null space of A has dimension n. **The dimension of the null space is $n - rank(A)$ which equals zero.**

TRUE (reduced row echelon form) `rref`(A) is the identity matrix. **From the Fundamental Theorem of Invertible Matrices, rref(A) is the identity matrix.**

FALSE There exists a nonzero square matrix B such that $AB = \mathcal{O}$, where $\mathcal{O}$ is the square zero matrix. $\mathbf{B} = \mathbf{A^{-1}}\mathcal{O} = \mathcal{O}$. B must be the zero matrix.

TRUE A^{-1} is an invertible matrix. **Since $(A^{-1})^{-1} = A$ we know that A and A^{-1} are both invertible.**

TRUE A^T is an invertible matrix. $(A^{-1})^T = (A^T)^{-1}$ **so since A^{-1} exists then A^T is invertible. Also determinanty of A and A^T are the same so they are both non-zero.**

FALSE There exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. **Repeat after me: "Invertibility and Diagonalizability are unrelated to each other!"**

FALSE One of the eigenvalues of A must be zero. **None of the eigenvalues of A can be zero, because the determinant equals the product of the eigenvalues and the determinant is non-zero so the product of eigenvalues must be non-zero, which means they are all non-zero.**

(b) To prove a statement is TRUE, it must always be true. **(c)** To prove a statement is FALSE, you just need to find a counter-example. Choose your statement to prove carefully. Note, the premise in each of these statements is that A is a square $n \times n$ invertible matrix, so your proofs must begin with that assumption.

#2 Subspaces, Orthogonal Complements. **(a)** The unnamed set $\mathcal{V}$ is a set of vectors $\vec{y}$ such that $\vec{y}A = \vec{0}$, i.e. the left nusllspace of A. In class we proved that the nullspace of A is a subspace of $\mathbb{R}^n$. In this case, null(A^T) is a subspace of $\mathbb{R}^m$ but this proof follows every proof for showing that a given set is a subpspace: answer three questions. (i) Is $\vec{0} \in \mathcal{V}$? Yes, because $\vec{0}A$ equals $\vec{0}$. (ii) Is $\mathcal{V}$ closed under scalar multiplication? In other words, if $\vec{y} \in \mathcal{V}$ then $c\vec{y} \in \mathcal{V}$? $(c\vec{y})A = c(\vec{y}A) = c\vec{0} = \vec{0}$ (iii) Is $\mathcal{V}$ closed under vector addition? In other words if $\vec{v}, \vec{w} \in \mathcal{V}$ then $(\vec{v} + \vec{w}) \in \mathcal{V}$. Yes, it is closed under vector addition. $\vec{v}A = \vec{0}$ and $\vec{w}A = \vec{0}$ so $\vec{v}A + \vec{w}A = (\vec{v} + \vec{w})A = \vec{0}$ **(b)** In order to find rank(A) you need to know rref(A). Since rref(A)$=\begin{bmatrix} 1 & 0 & 1 & 5/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ It turns out that rank(A)=2. Once you know rank, immediately you know the dimension of the row space and the column space are equal to the rank=2, while the dimension of the nullspace is n (number of columns) minus the rank=4-2=2 and the dimension of the left nullspace is m (the number of rows) minus the rank=3-2=1. **(c)** To find bases for $\mathcal{V}$ and $\mathcal{V}^\perp$ one needs to know which subspaces they correspond to. $\mathcal{V} = $ null(A^T) and $\mathcal{V}^\perp = $ col(A) so a basis for column space can be found from rref as $\{(1, 0, -3), (-1, 2, 17)\}$ and a basis for its orthgonal complement is $\{(3, -7, 1)\}$ which can be computed from rref(A^T) or using the cross-product.

#3 Eigenvalues, Determinants, Diagonalization, Grams-Schmidt Orthogonalization, Orthogonal Matrices, Bases. **(a)** In order to compute A^{100} you need to find the eigenvalues and eigenvectors of A.
$$A^{100} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2-2^{100} & -2+2^{101} \\ 1-2^{100} & -1+2^{101} \end{bmatrix}.$$

An orthogonal matrix has columns which are orthonormal to each other. Starting with $(1/\sqrt{2}, 0, 1/\sqrt{2})$ one mayb be able to see that $(0,1,0)$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})$ to make an orthonormal set in $\mathbb{R}^3$. So $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$ and multiplying QQ^T or Q^TQ produces the identity matrix $\mathcal{I}$. Otherwise, you can use $\left\{(1/\sqrt{2}, 0, 1/\sqrt{2}), (0,1,0), (1,0,0)\right\}$ as your basis for $\mathbb{R}^3$ but then, starting with $\vec{a} = (1/\sqrt{2}, 0, 1/\sqrt{2})$ use Gram-Schmidt orthogonalization to convert this non-orthogonal basis into an orthonormal basis for $\mathbb{R}^3$ which become the columns of an orthogonal Q matrix.