# Test 2: LINEAR SYSTEMS

Math 214 Spring 2008 | Friday April 18
©Prof. Ron Buckmire | 9:30pm-10:25pm

Name: KEY

**Directions**: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

**Pledge:** I, ________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

| No. | Score | Maximum |
|---|---|---|
| 1 | | 30 |
| 2 | | 40 |
| 3 | | 30 |
| BONUS | | 10 |
| **Total** | | **100** |

**1. Invertible Matrices.** *30 points.*

Suppose $A$ is an **invertible** $n \times n$ matrix, where $n > 1$.

(a) (*20 points.*) Determine whether each of the following is true or false. Just write T or F in front of each, without explanation.

F The equation $A\vec{x} = \vec{0}$ has a nonzero solution.

T For every vector $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ always has exactly one solution.

T The rows of $A$ form a basis for $\mathbb{R}^n$.

F The null space of $A$ has dimension $n$.

T (reduced row echelon form) rref(A) is the identity matrix.

F There exists a nonzero square matrix $B$ such that $AB = \mathcal{O}$, where $\mathcal{O}$ is the square zero matrix.

★ T $A^{-1}$ is an invertible matrix.

T $A^T$ is an invertible matrix.

F There exists an invertible matrix $P$ such that $P^{-1}AP = D$ where $D$ is a diagonal matrix.

★ F One of the eigenvalues of $A$ must be zero.

(b) (*5 points.*) Choose one of the statements in part (a) that you determined is TRUE and **prove** that statement is true.

"$A^{-1}$ is an invertible matrix." TRUE.

$AA^{-1} = I$, $A^{-1}A = I$ $\Rightarrow$ $A = (A^{-1})^{-1}$ So $(A^{-1})^{-1}$ exists and equals $A$ (which is invertible

#1. ~~A is invertible $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$. $\vec{b}$ is only solution~~

(c) (*5 points.*) Choose one of the statements in part (a) that you determined is FALSE and **prove** that statement is false.

"One of the eigenvalues of A must be zero." FALSE.

$\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0$ since $A$ is invertible

Since the product of the eigenvalues is non-zero, none of them are zero.

**2. Subspaces, Orthogonal Complements.** *40 points.*

(a) (*12 points.*) Suppose that $A$ is an $m \times n$ matrix and $\mathcal{V} = \left\{ \text{The set of vectors } \vec{y} \text{ such that } \vec{y}A = \vec{0} \right\}$. Is $\mathcal{V}$ a subspace of a vector space? PROVE YOUR ANSWER. (NOTE: You do not have to prove $\mathcal{V}$ is a vector space, just a subspace.)

Yes $\mathcal{V}$ is a subspace, it is the left nullspace of A.

① Is $\vec{0} \in \mathcal{V}$? YES!

If $\vec{y} = 0 \Rightarrow \vec{0}A = \vec{0}$ so 0 is in $\mathcal{V}$.

② Is $\mathcal{V}$ closed under scalar multiplication? YES!

$\vec{y}A = \vec{0} \Rightarrow (c\vec{y})A = c(\vec{y}A) = c\vec{0} = \vec{0}$

This means $c\vec{y} \in \mathcal{V}$

③ Is $\mathcal{V}$ closed under vector addition? Yes!

$\vec{y}A = \vec{0}$ and $\vec{x}A = \vec{0} \Rightarrow \vec{x}A + \vec{y}A = 0$

$(\vec{x} + \vec{y})A = 0$

Thus $(\vec{x} + \vec{y}) \in \mathcal{V}$

(b) (*12 points.*) Suppose $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ -3 & 17 & -20 & -2 \end{bmatrix}$. What is the rank of $A$? Use this information to write down the dimensions of all the four associated subspaces of the matrix $A$. EXPLAIN YOUR ANSWER.

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ -3 & 17 & -20 & -2 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_1} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 14 & -14 & -7 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{7}R_3} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

$$\xrightarrow[R_2' \to \frac{1}{2}R_2]{R_3' \to R_3 - R_2} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1' \to R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 7/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

r = rank(A) = 2 = dim col(A) = dim row(A)

null(A) = 4 − 2 = n − r = 2

null($A^T$) = 3 − 2 = m − r = 1

(c) (*12 points.*) Again consider $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ -3 & 17 & -20 & -2 \end{bmatrix}$. Find bases for $\mathcal{V}$ and its orthogonal complement $\mathcal{V}^\perp$ as defined in part (a). **SHOW ALL YOUR WORK.**

$\mathcal{V}$ is the leftnull space and $\mathcal{V}^\perp$ is its complement the column space

A basis for the column space is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 17 \end{pmatrix} \right\}$

$\mathcal{V}^\perp = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 17 \end{pmatrix} \right\}$

A basis for the left null space is orthogonal to these two vectors.

(i) either find rref($A^T$) or

(ii) Take cross product of $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 17 \end{pmatrix}$

(ii) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ -1 & 2 & 17 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & -3 \\ 2 & 17 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -3 \\ -1 & 17 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 6\hat{i} - 14\hat{j} + 2\hat{k} = \begin{pmatrix} 6 \\ -14 \\ 2 \end{pmatrix}$

~~At~~ $\mathcal{V} = \text{span}\left\{ \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix} \right\}$

(i) $A^T = \begin{pmatrix} 1 & 0 & -3 \\ -1 & 2 & 17 \\ 2 & -2 & -20 \\ 3 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 14 \\ 0 & -2 & -14 \\ 0 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Strang method $\text{null}(A^T) = \text{span}\begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix}$

(d) (*4 points.*) Use your bases for $\mathcal{V}$ and $\mathcal{V}^\perp$ to verify they are indeed orthogonal complements.

$\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix} = 3 + 0 - 3 = 0$ ✓

$\begin{pmatrix} -1 \\ 2 \\ 17 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix} = -3 - 14 + 17 = 0$ ✓

The bases are orthogonal to each other

**3. Eigenvalues, Determinants, Diagonalization, Gram-Schmidt Orthogonalization, Orthogonal Matrices, Bases.** *30 points.*

Do only one of the following problems:

Let $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$. Find $A^{100}$. SHOW ALL WORK. (HINT: diagonalize the matrix $A$; your answer should be a single $2 \times 2$ matrix).

**OR**

Find an orthogonal matrix $Q$ which has the vector $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ as one of its columns.

SHOW ALL WORK. (HINT: use the given vector plus two more of your choice to produce an orthonormal basis for $\mathbb{R}^3$. To ease your calculations, choose vectors that have at least one zero component.) **Verify** the matrix you find is indeed orthogonal by computing $QQ^T$.

$p(\lambda) = \lambda^2 - 3\lambda + 2 = 0 = (\lambda-2)(\lambda-1) = 0 \Rightarrow \lambda = 1, 2$

$\text{null}(A - I) = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ $\qquad$ $\text{null}(A - 2I) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\qquad$ $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^{100} & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2^{100} \\ 1 & 2^{100} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2-2^{100} & -2+2^{101} \\ 1-2^{100} & -1+2^{101} \end{pmatrix}$$

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A orthogonal basis for $\mathbb{R}^3$ with $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

orthonormal basis for $\mathbb{R}^3$ is $\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = Q = Q^T$

$$Q^TQ = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & 0 & \frac{1}{2}-\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2}+\frac{1}{2} & 0 & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= I$

**BONUS QUESTION.** (*10 points.*)
Consider $\mathcal{V}$ and $\mathcal{V}^\perp$ from Question 2. Can the vector $(2,1,1)$ be expressed as a sum of two non-zero vectors $\vec{v} \in \mathcal{V}$ and $\vec{v}^\perp \in \mathcal{V}^\perp$? Find the values of $\vec{v}$ and $\vec{v}^\perp$ or explain why they do not exist.

OR

Prove that IF $\lambda$ is an eigenvalue of a square matrix $A$, THEN $\det(A - \lambda\mathcal{I}) = 0$.

$$\begin{pmatrix}2\\1\\1\end{pmatrix} = \operatorname{proj}_{\mathcal{V}}\begin{pmatrix}2\\1\\1\end{pmatrix} + \operatorname{proj}_{\mathcal{V}^\perp}\begin{pmatrix}2\\1\\1\end{pmatrix} = \underbrace{\frac{\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-3\end{pmatrix}}{\begin{pmatrix}1\\0\\-3\end{pmatrix}\begin{pmatrix}1\\0\\-3\end{pmatrix}}\begin{pmatrix}1\\0\\-3\end{pmatrix} + \frac{\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\17\end{pmatrix}}{\begin{pmatrix}-1\\2\\17\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\17\end{pmatrix}}\begin{pmatrix}-1\\2\\17\end{pmatrix}}_{\operatorname{proj}_{\mathcal{V}^\perp}(\vec{b})}$$

$$+ \underbrace{\frac{\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-7\\1\end{pmatrix}}{\begin{pmatrix}3\\-7\\1\end{pmatrix}\begin{pmatrix}3\\-7\\1\end{pmatrix}}\begin{pmatrix}3\\-7\\1\end{pmatrix}}_{\operatorname{proj}_{\mathcal{V}}(\vec{b})}$$

$$\text{BUT } \begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-7\\1\end{pmatrix} = 6-7+1=0$$

This means $\begin{pmatrix}2\\1\\1\end{pmatrix}$ lies IN $\mathcal{V}^\perp$ and CAN NOT be written as a NON ZERO ~~~~ SUM of vectors from $\mathcal{V}$ and $\mathcal{V}^\perp$.

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PROOF: $\lambda$ is an eigenvalue of $A$. ~~For~~ each eigenvalue there is atleast one associated eigenvector $\vec{x}$

Thus $A\vec{x} = \lambda\vec{x}$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$(A-\lambda I)\vec{x} = \vec{0}$$
$$B\vec{x} = \vec{0}$$

(Let $B = A - \lambda I$)

By the Fund Thm of Inv Matrices ~~If~~ $\det(B) = 0 \Leftrightarrow B$ is not invertible

~~If~~ $B$ is not invertible $\Leftrightarrow$ null$(B)$ is NOT empty

$\therefore$ null$(B)$ not empty $\Leftrightarrow \det(B) = 0$

null$(B)$ equals the eigenspace of $A$ which we know is not empty so $\det(A - \lambda I) = 0$.