Linear Systems

Math 214 Spring 2007 — Fowler 110 MWF 2:30pm - 3:25pm

 — http://faculty.oxy.edu/ron/math/214/07/

Class 28: **Monday April 9**

TITLE Gram-Schmidt Process and QR factorization
CURRENT READING Poole 5.3

Summary
There is a very cool algorithm for producing an orthogonal basis from "regular" basis for a given subspace. The process is known as Gram-Schmidt orthogonalization. We also are introduced to *another* matrix factorization, $A = QR$.

Homework Assignment
*HW #26 Poole, Section 5.3 : 1,***2,3***,4,***6,11,13***, 17. EXTRA CREDIT 18.*

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1. Gram-Schmidt Orthogonalization

Suppose we start off with three linearly independent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. First we will construct three orthogonal vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and then normalize these to produce three orthonormal vectors $\vec{q_1}$, $\vec{q_2}$ and $\vec{q_3}$ from our original linearly independent trio.

STEP 1. First choice, start with $\vec{a}$.
1. Let $\vec{A} = \vec{a}$.

STEP 2. Second choice, select in the direction of $\vec{b}$ with the projection in the direction of $\vec{a}$ **removed**. Then this vector should be orthogonal to $\vec{a}$.

2. Let $\vec{B} = \vec{b} - \dfrac{\vec{A}^T\vec{b}}{\vec{A}^T\vec{A}}\vec{A} = \vec{b} - \text{proj}_{\vec{A}}(\vec{b}) = \text{perp}_{\vec{A}}(\vec{b})$

STEP 3. Third choice, select in the direction of $\vec{c}$ with the projections of $\vec{c}$ in the direction of $\vec{a}$ and in the direction of $\vec{b}$ removed. So this third vector will be orthogonal to both of those!

3. Let $\vec{C} = \vec{c} - \dfrac{\vec{A}^T\vec{c}}{\vec{A}^T\vec{A}}\vec{A} - \dfrac{\vec{B}^T\vec{c}}{\vec{B}^T\vec{B}}\vec{B} = \vec{c} - \text{proj}_{\vec{A}}(\vec{c}) - \text{proj}_{\vec{B}}(\vec{c})$

STEP 4. Normalize A, B and C by dividing by their magnitudes to obtain $\vec{q_1} = \frac{\vec{A}}{||\vec{A}||}$, $\vec{q_2} = \frac{\vec{B}}{||\vec{B}||}$ and $\vec{q_3} = \frac{\vec{C}}{||\vec{C}||}$.
The vectors $\vec{q_1}$, $\vec{q_2}$ and $\vec{q_3}$ are **orthonormal**!

EXAMPLE

Let's use Gram-Schmidt to convert the linearly independent vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}$ to three orthonormal vectors.

2. A=QR Factorization

Gram-Schmidt Orthogonalization is equivalent to factoring a $m \times n$ matrix A into the product of a matrix Q with orthonormal columns and R is an invertible upper triangular matrix.

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{q_1} & \vec{q_2} & \vec{q_3} \end{bmatrix} \begin{bmatrix} \vec{q_1}^T\vec{a} & \vec{q_1}^T\vec{b} & \vec{q_1}^T\vec{c} \\ & \vec{q_2}^T\vec{b} & \vec{q_2}^T\vec{c} \\ & & \vec{q_3}^T\vec{c} \end{bmatrix}$$

Exercise

Strang, page 230, #23. Find $\vec{q_1}$, $\vec{q_2}$, and $\vec{q_3}$ as combinations of the independent columns of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$ and write A as QR.