Linear Systems

Math 214 Spring 2007 — Fowler 110 MWF 2:30pm - 3:25pm

http://faculty.oxy.edu/ron/math/214/07/

Class 25: **Monday April 2**

TITLE Computational Techniques for Computing Eigenvalues
CURRENT READING Poole 4.5

Summary
In practice one often computes the eigenvalues of a matrix using something known as the "power method." We will explore this method and some interesting features of eigenvalues.

Homework Assignment
*HW#23 Poole, Section 4.5 : 3,4,5,***6,10,13,15***, 21, 25,***47, 54***. EXTRA CREDIT 45*

DEFINITION

A **dominant eigenvalue** of a $n \times n$ matrix is an eigenvalue which is strictly greater in absolute value or magnitude than all the other eigenvalues of the matrix. The eigenvector corresponding to the dominant eigenvalue is called the **dominant eigenvector**.

Theorem 4.28

Let A be a diagonalizable $n \times n$ matrix with dominant eigenvalue λ^*. THEN there exists a non-zero vector $\vec{x}_0$ such that the sequence of vectors $\vec{x}_k$ defined by

$$\vec{x}_1 = A\vec{x}_0, \quad \vec{x}_2 = A\vec{x}_1, \quad \vec{x}_3 = A\vec{x}_2, \quad \ldots, \quad \vec{x}_k = A\vec{x}_{k-1}$$

approaches a dominant eigenvector of A.

GroupWork

Let's try to use this method to approximate the dominant eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ by having each group pick an initial vector of your choice, like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

You should fill in the table below with the current value of $\vec{x}_k$, l_k (the ratio of the first component of $\vec{x}_{k+1}$ to the first component of $\vec{x}_k$) and r_k (the ratio of the first component of $\vec{x}_k$ to the second component of $\vec{x}_k$)

k	0	1	2	3	4	5	6	7	8	9	10	11	...
$\vec{x}_k$													
l_k	XXX												
r_k	XXX												

What do your results tell you are the dominant eigenvector and dominant eigenvalue of the matrix A?

1. The Power Method

The **algorithm** (sequence of steps) which leads to the computation of the dominant eigenvalue and dominant eigenvector is called **The Power Method**.

ALGORITHM

Let A be a diagonalizable $n \times n$ matrix with dominant eigenvalue λ^*.
1. Let $\vec{x}_0 = \vec{y}_0$ be an initial vector in $\mathbb{R}^n$ with largest component of size 1.
2. Repeat the following steps for $k = 1, 2, 3, \ldots$

(a) Compute $\vec{x}_k = A\vec{y}_{k-1}$.

(b) Let m_k be the component of $\vec{x}_k$ with the largest absolute value.

(c) Set $\vec{y}_k = (1/m_k)\vec{x}_k$.

For most choices of $\vec{x}_0$, m_k converges to the dominant eigenvalue λ^* and $\vec{y}_k$ converges to the dominant eigenvector.

EXAMPLE

Let's use the power method (and MATLAB) to compute the dominant eigenvector and dominant eigenvalue of $A = \begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$

DEFINITION

Let A be a real or complex $n \times n$ matrix and let r_i denote the sum of the absolute values of the off-diagonal entries in the i^{th} row of A; that is $r_i = \sum_{j \neq i}^{n} |a_i j|$. The i^{th} **Gerschgorin disk** is the circular disk D_i in the complex plane with center a_{ii} and radius r_i. That is, $D_i = \{z \text{ in } \mathbb{C} : |z - a_{ii}| \leq r_i\}$

Exercise

Sketch the Gerschgorin disk and the eigenvalues for the following matrices:

(a) $\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$

Theorem 4.29

Gerschgorin's Theorem. Let A be a $n \times n$ (real or complex) matrix. THEN every eigenvalue of A is contained within a Gerschgorin disk.