Linear Systems

Class 24: **Friday March 30**

TITLE Diagonalization and Similarity
CURRENT READING Poole 4.4

Summary
One application of computing eigenvalues and eigenvectors leads to an important matrix factorization and characteristic of a matrix known as "diagonalizability."

Homework Assignment
HW#22: Poole, Section 4.4: **2**,*5*,*6*, **9, 10, 16,18**,*21*,*22*,*24*,*25. EXTRA CREDIT 23.*

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1. Factoring A $= S\Lambda S^{-1}$

S is a matrix whose columns consist of the eigenvectors of A.
Λ is a diagonal matrix with the eigenvalues of A along the diagonal.
The factorization is only possible if the $n \times n$ (square) matrix A has exactly n *linearly independent* eigenvectors. In other words, none of the eigenvectors can be a linear combination of the other eigenvectors (other wise S^{-1} would not exist).
Let's show that $A = S\Lambda S^{-1}$ and $AS = S\Lambda$ **and** $\Lambda = S^{-1}AS$. This last form is the most important, because it means that we can produce a diagonal matrix Λ from a given square matrix A by pre- and post- multiplying it by the special matrix S. This process is called **diagonal decomposition**.

Proof
If $\vec{x_1}, \vec{x_2}, \vec{x_3}, \ldots, \vec{x_n}$ are n linearly independent eigenvectors of A which make up the columns of a special matrix S then

$$\begin{aligned} AS &= A\begin{bmatrix} \vec{x_1} & \vec{x_2} & \vec{x_3} & \ldots & \vec{x_n} \end{bmatrix} = \begin{bmatrix} A\vec{x_1} & A\vec{x_2} & A\vec{x_3} & \ldots & A\vec{x_n} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\vec{x_1} & \lambda_2\vec{x_2} & \lambda_3\vec{x_3} & \ldots & \lambda_n\vec{x_n} \end{bmatrix} = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \vec{x_3} & \ldots & \vec{x_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} = S\Lambda \end{aligned}$$

The diagonalization matrix factorization $A = S\Lambda S^{-1}$ is a special case of **similar matrices**.

DEFINITION
A is said to be **similar** to B if there exists an invertible $n \times n$ matrix P so that $B = P^{-1}AP$ (and thus $PB = AP$ or $AP = PB$). If A is similar to B we say that $A \sim B$.

The process of diagonalization is finding a diagonal matrix which is similar to the given $n \times n$ matrix A.

EXAMPLE Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Similar Matrices

Theorem 4.21

Let A, B and C be $n \times n$ matrices.
(i) $A \sim A$ (**Reflexivity**)
(ii) If $A \sim B$, then $B \sim A$ (**Symmetry**)
(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ (**Transitivity**)
You'll see more about these words (reflexive, symmetric and transitive) in **Math 210**! If a relation $\sim$ satisfies these properties it is known as an **equivalence relation**.

Exercise Can you prove each of the results in Theorem 4.21? You should be able to!

Theorem 4.22

Let A and B be two similar $n \times n$ matrices. THEN
(a) $\det(A) = \det(B)$
(b) A is invertible if and only if B is invertible.
(c) A and B have the same rank.
(d) A and B have the same characteristic polynomial.
(e) A and B have the same eigenvalues.

EXAMPLE Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let's show that $A$and B have the same characteristic polynomial, the same eigenvalues, are both invertible, have rank 2 and the same determinant.

Q: Are these two matrices A and B similar to each other?
A: No! Does this mean that Theorem 4.22 is a vicious lie? Explain the apparent contradiction.

3. Matrix Exponentiation One useful result of diagonal decomposition is that it allows us to compute values of A^n very easily. It is very easy to exponentiate a diagonal matrix.

$A^{10} = (S\Lambda S^{-1})^{10} = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})$

Can we simplify this expression? YES!

$A^{10} = S\Lambda^{10}S^{-1}$

EXAMPLE

Compute $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^{10}$

4. More on Diagonalization

Theorem 4.25

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Theorem 4.26

The geometric multiplicity (the dimension of the eigenspace) of each eigenvalue is always less than of equal to the algebraic multiplicity (the multiplicity of the eigenvalue as a root of the characteristic polynomial).

Theorem 4.27

Let A be an $n \times n$ matrix with k distinct eigenvalues. The following statements are equivalent:
(a) A is diagonalizable.
(b) The union β of the bases of the eigenspaces of A contains n vectors.
(c) The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

GROUPWORK

Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. Are either of these matrices diagonalizable? Explain your answer!

One application of matrix diagonalization is the computation of the matrix exponential, e^A. Similar to the definition of $A^n = S\Lambda^n S^{-1}$, if A is diagonalizable, then it has n linearly independent eigenvectors to make up the columns of S and thus

$$e^A = S \begin{bmatrix} e^{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n} \end{bmatrix} S^{-1}$$

EXAMPLE

Let's compute e^A, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

5. Symmetric matrices are always diagonalizable

Consider the matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & d \end{bmatrix}$. Show that it has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ d-1 \end{bmatrix}$ with eigenvalues $-1, 1, d$ respectively.

When $d \to 1$ the third eigenvector (and eigenvalue) collapses to be the same as the second, so that the S matrix for A will be singular and thus A will not be diagonalizable.

However, now consider the symmetric matrix $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & d \end{bmatrix}$. Show that it has eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with eigenvalues $-1, 1, d$ respectively.

As $d \to 1$ the second eigenvalue repeats, but the eigenvectors are unaffected. Note again: The eigenvectors are perpendicular (i.e. orthogonal) to each other so the matrix B can be diagonalized. The S matrix of eigenvectors will be non-singular and thus S^{-1} will exist. **Do it!**