Report on Test 1

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Point Distribution (N=13)

Range	99+	92+	89+	87+	82+	80+	77+	73+	70+	65+	60+	55+	55-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	3	0	1	0	4	0	2	2	0	1	0	0	0

Summary Overall class performance was somewhat bifurcated. The mean, median and modal score was 85. The high score was a 103. The low score was 68.

#1 Span, Linear Independence, Rank. This is a short answer question. It is asking about the implications of rref$(A) \neq \mathcal{I}$. **(a)** rref$\left(\begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ **(b)** The rank of A is the number of non-zero rows in rref(A), which in this case is 1. **(c)** The span of the columns of A is the set of all possible linear combinations of the columns. Since $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 6 \\ 18 \end{bmatrix}$ and they are both multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, the span of the columns is the set consisting of all multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Represnted graphically, this object is the line $y = 3x$. **(d)** Since the columns are scalar multiple of each other, and since the rank of A is less than the number of rows in the matrix, the columns are linearly DEPENDENT. **(e)** Since rref$(A) \neq \mathcal{I}$ by the Fundamental Theorem of Invertible Matrices, A^{-1} does not exist. Also, the determinant of A is $(3)(18) - (9)(6) = 54 - 54 = 0$ which also implies that the matrix is NOT invertible.

#2 Row reduction, Reduced Row Echelon Form, Identity, Invertibility. This question is about thinking about when a given matrix can be transformed using elementary row operations into any another matrix. **(a)** $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & 2 & 6 \end{bmatrix}$ is transformed into $\mathcal{I}_3$ **(b)** $\mathcal{I}_3$ is transformed to $B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **(c)** Since it is clear that $A \to \mathcal{I}_3$ and $\mathcal{I}_3 \to B$ it follows that $A \to B$. Since A can be transformed into the identity, there must exist an inverse A^{-1}, so A is invertible. Using elementary matrices $(E_k \ldots E_1)A = I \Rightarrow A = (E_k \ldots E_1)^{-1}$ and $A^{-1} = E_k \ldots E_1$. Similarly $(F_k \ldots F_1)I = B \Rightarrow I = (F_k \ldots F_1)^{-1}B$ which implies that B^{-1} exists and equals $(F_k \ldots F_1)^{-1}$. The overall principle here is that any invertible matrix can be transformed into any OTHER invertible matrix using a sequence of elementary row operations.

#3 Matrix Operations, Trace, Transpose. Recall this is a TRUE or FALSE question. For FALSE, all you have to do is find one counter example. For TRUE, you have to prove the given statement is always TRUE. **(a)** "tr(I_n)=n" **TRUE.** Since the $n \times n$ identity matrix has n rows with one 1 along the diagonal, its trace (which is the sum of the diagonal elements) will equal 1 added n times, i.e. tr(I_n)=n. **(b) FALSE.** "tr(A)tr(B)=tr(AB)." It's fairly straightforward to come up with a counter example for this statement, especially where A is either the identity matrix. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. tr(AB) $\neq$ tr(A)tr(B) in this case. **(c)** "tr$(AA^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2$." **TRUE.** This is a pretty cool result I stumbled upon. Although the question limits itself to square matrices, it is true for all $m \times n$ matrices. Recall that the ij^{th} element of a matrix product AB is formed by taking the dot product of the i^{th} row of A with the j^{th} column of B. Thus the i^{th} diagonal element of AA^T, $(AA^T)_{ii} = \text{row}_i(A) \cdot \text{col}_i(A^T) = \text{row}_i(A) \cdot \text{row}_i(A) =$ sum of the square of the elements in the i^{th} row. (Recall columns of A^T equal rows of A.) Thus the trace of the product AA^T will equal the sum of the square of all the elements from every row.

#4 Linear Systems, Equations of Planes and Lines. Whenever you see a linear system the first thing you should check is to see if it is homogeneous or not. (Because if it has the form $A\vec{x} = \vec{0}$ then you know it must have 1 or ∞ solutions.) Second thing is to put it in augmented matrix form and find rref(A). **(a)** The given system is homogeneous, so it must have at least the trivial solution. $\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ This means that $3x - 4z = 0$, $y = 0$ and $0 = 0$. There are an infinite number of non-trivial solutions since you have a zero row in rref(A). **(b)** The solution set is $x = \frac{4}{3}t, y = 0, z = t$ where t can be any real number. This corresponds to the parametric equations for a line, since there is 1 free variable it is describing a one-dimensional object. The vector equation form is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$. **(c)** Since the system is homogeneous the object representing the solution set must pass through the origin (0,0,0) thus the minimum distance from the origin (of SOLUTIONS OF ALL HOMOGENEOUS SYSTEMS) is always zero.

 # BONUS Analytic Geometry, Projections. This question refers to finding the minimum distance between a line with a given direction vector and a given point. The line is $\vec{x} = t\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and the point is $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The simplest way to do the problem is to pick a point on the line and find the difference between a point on the line and the given point to determine the position vector $\vec{p}$ from the line to the given point. Then find the projection of this position vector in the direction of the direction vector of the line. The difference between the position vector and the projection vector will be the orthogonal distance vector between a point on the line and the point in question. Since the line goes through the origin, let the position vector be $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the direction vector $\vec{d} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$. The distance vector $\vec{D} = \vec{p} - \text{proj}_{\vec{d}}(\vec{p})$.

$$\begin{aligned}
\vec{D} &= \vec{p} - \text{proj}_{\vec{d}}(\vec{p}) \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{7}{25}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{28}{25} \\ 0 \\ \frac{21}{25} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-3}{25} \\ 1 \\ \frac{4}{25} \end{bmatrix}
\end{aligned}$$

The minimum distance will be the magnitude of this vector, which is $\sqrt{\left(\frac{-3}{25}\right)^2 + 1^2 + \left(\frac{4}{25}\right)^2} = \frac{\sqrt{26}}{5}$.