FINAL EXAM: Linear Systems

Monday May 14, 2007: 8:30-11:30am — Math 214

Name: Key

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Directions: Read *all* problems first before answering any of them. There are EIGHT (8) problems. They are NOT related. The first four problems involve proofs and are more theoretical, the last four problems are more calculation oriented.

This exam is a closed-notes, closed-book, test. No calculators.

You must include ALL relevant work to support your answers. Use complete English sentences where possible and CLEARLY indicate your final answer from your "scratch work.'

Pledge: I, ____________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1.		**25**
2.		**25**
3.		**25**
4.		**25**
5.		**25**
6		**25**
7.		**25**
8.		**25**
TOTAL		**200**

1. *[25 points total.]* **TRUE or FALSE**. TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE.

(a) If $\vec{x}$ and $\vec{y}$ are orthogonal, then they are linearly independent.

$\boxed{\text{TRUE}}$

$\vec{x} \cdot \vec{y} = 0$

(Proof by contradiction)

If $\vec{x}$ and $\vec{y}$ were lin dependent (and $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$)
then $\vec{x} = c\vec{y}$ (where $c \neq 0$)

$(c\vec{y}) \cdot \vec{y} = 0$

$c(\vec{y} \cdot \vec{y}) = 0$

$c|\vec{y}|^2 = 0$

only true if $\vec{y} = 0$,
which is a contradiction.

Thus $\vec{x}$ and $\vec{y}$ are NOT lin dependent, or they are lin independent

(b) The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$\boxed{\text{FALSE}}$

row(A) = span of rows of A (which have n components each)

row(A) $\subset \mathbb{R}^n$

row(A) $\subset \mathbb{R}^3$ not $\mathbb{R}^2$

$A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$

(c) If A is a symmetric matrix then $A + I$ is also a symmetric matrix.

$\boxed{\text{TRUE}}$

$$(A+B)^T = A^T + B^T$$

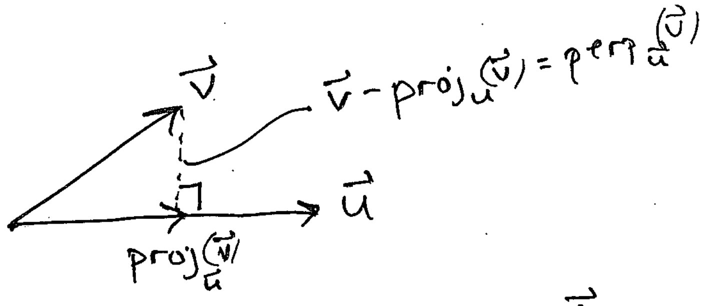
$$\begin{aligned}(A+I)^T &= A^T + I^T \\ &= A + I^T \quad &\text{(since } A \text{ is symmetric)} \\ &= A + I \quad &\text{(since } I \text{ is symmetric)}\end{aligned}$$

(d) $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = \vec{0}$, where $\text{proj}_{\vec{u}}(\vec{v}) = \left(\dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$. (Draw a picture!)

$\boxed{\text{TRUE}}$

$\text{proj}_u(\text{perp}_u(\vec v)) = 0$

$$= \frac{\vec u \cdot \left(\vec v - \left(\frac{\vec v \cdot \vec u}{\vec u \cdot \vec u}\right)\vec u\right)}{\vec u \cdot \vec u}(\vec u)$$



$$= \left(\frac{\vec u \cdot \vec v - \frac{\vec v \cdot \vec u (\vec u \cdot \vec u)}{(\vec u \cdot \vec u)}}{\vec u \cdot \vec u}\right)\vec u = \left(\frac{\vec u \cdot \vec v - \vec v \cdot \vec u}{\vec u \cdot \vec u}\right)\vec u = 0\vec u = \vec 0$$

(e) The set of solution vectors to the linear system $A\vec{x} = \vec{b}$ is a vector space.

$\boxed{\text{FALSE}}$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow \vec x = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \leftarrow$$

This is a single point in space not including $\vec 0$, so it is NOT a vector space

2. *[25 points total.]* **Linear Systems, Column Space. Solvability.**

a. (*15 points*). Prove the statement(s) "The linear system $A\vec{x} = \vec{b}$ has no solution IF AND ONLY IF $\vec{b}$ is not in the column space of the $m \times n$ matrix A."

$\Rightarrow$ $A\vec{x} = \vec{b}$ has no solution means $\vec{x}$ does NOT exist such that a linear combination of the columns of A, i.e. $A\vec{x}$, will equal $\vec{b}$, thus $\vec{b}$ is NOT in the column space of A.

$\Leftarrow$ If $\vec{b} \notin col(A)$, this means $\vec{b}$ is NOT a linear combination of the columns of A which means

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{n2} \end{pmatrix} + \dots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \neq \vec{b}$$

this means $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \vec{b}$

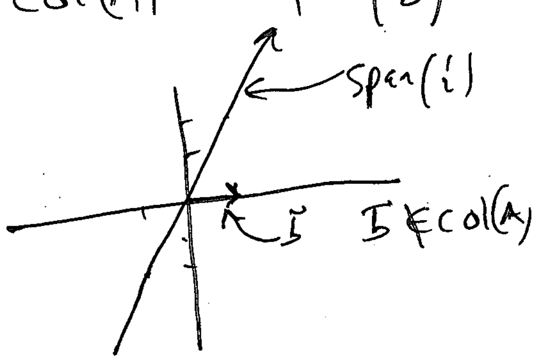
this means $A\vec{x} \neq \vec{b}$ this means there is no solution to $A\vec{x} = \vec{b}$.

b. (*10 points*). Provide an example of a linear system $A\vec{x} = \vec{b}$ and the column space of A and also show that $\vec{b} \notin \text{col}(A)$ and your given linear system $A\vec{x} = \vec{b}$ has no solutions.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \end{array}\right)$$

$0 \neq -1$ implies no solution

$$col(A) = span\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



3. *[25 points total.]* **Linear Independence.**

a. (*10 points*). **If a matrix has more rows than columns, then its columns must be linearly dependent.** Prove this statement is either TRUE or FALSE.

$$\underbrace{\left.\begin{pmatrix} & m<n & \end{pmatrix}\right\}}_{n \text{ columns}} m \text{ rows} = A.$$

The statement is TRUE

(1) If $m<n$ then the largest rank(A) could be is m, which means $\operatorname{rank}(A) < n$, so $\dim(\operatorname{null}(A)) = n - \operatorname{rank}(A) > 0$.

If the nullspace is <u>NOT</u> empty then we know the columns of A must be linearly dependent.

(2) Another reason is you have n vectors in $\mathbb{R}^m$ where $n>m$, thus you know that these vectors must be lin dependent since it only take m vectors to span $\mathbb{R}^m$

b. (*10 points*). **If a matrix has more columns than rows, then its columns must be linearly dependent.** Prove this statement is either TRUE or FALSE.

$$m \left\{ \underbrace{\begin{pmatrix} & m>n & \end{pmatrix}}_{n} \right.$$

The statement is FALSE.

If $m>n$ the largest rank(A) can be is n.

In that case, $\dim \operatorname{null}(A) = n-n=0$, or the nullspace is empty and thus the columns are linearly independent

CounterExample:

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \\ 3 & 11 \end{pmatrix} \quad \begin{matrix} m=3 \\ n=2 \end{matrix} \quad m>n$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 4 \\ 8 \\ 11 \end{pmatrix}$ so the columns are lin independent even though there are more rows than columns

c. (*5 points*). Write down an example of a matrix which has more rows than columns and another matrix that has more columns than rows which go along with your answers in part (a) and (b)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$m=2$, $n=3$, $3>2$

Columns are lin dependent

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \\ 3 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

4. *[25 points total.]* **Vector Spaces.**

a. (*15 points*). Prove that the left nullspace, null(A^T), i.e. the set of solution vectors to $A^T\vec{x} = \vec{0}$ (or $\vec{x}^T A = \vec{0}$), is a subspace.

Prove V is a subspace, it must

(1) Contain $\vec{0}$

$A^T\vec{0} = \vec{0}$, i.e. $\vec{0}$ is a solution to $A^T\vec{x} = \vec{0}$,

so $\vec{0} \in \text{null}(A^T)$

(2) Be closed under scalar multiplication

$A^T(c\vec{x}) = c(A^T\vec{x}) = c(\vec{0}) = \vec{0}$

Thus $c\vec{x} \in \text{null}(A^T)$ when $\vec{x} \in \text{null}(A^T)$

(3) Be closed under vector addition

$A^T(\vec{x}+\vec{y}) = A^T\vec{x} + A^T\vec{y} = \vec{0} + \vec{0} = \vec{0}$

Thus $\vec{x}+\vec{y} \in \text{null}(A^T)$ when $\vec{x} \in \text{null}(A^T)$ and $\vec{y} \in \text{null}(A^T)$

b. (*10 points*). Find the left null space of $\begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 4 & 7 \end{bmatrix}$. What vector space is the left nullspace a subspace of? What vector space is it orthogonal to?

A is 3×2

$A^T\vec{x} = \vec{0} \Leftrightarrow \begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 \\ 0 & -13 & -13 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$x + z = 0 \Rightarrow x = -z$

$y + z = 0 \quad y = -z$

$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\text{null}(A^T) = \text{span}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\text{null}(A^T) \subset \mathbb{R}^3$

$\text{null}(A^T)$ is orthogonal to $\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}\right)$

If so, basis vectors will be orthogonal

$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = -1 - 3 + 4 = 0$ ✓

$\begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -5 - 2 + 7 = 0$ ✓

5. *[25 points total.]* **Determinants, Block Matrices.**
Consider the 4×4 matrix

$$M = \begin{bmatrix} 0 & a & b & d \\ -a & a & c & e \\ -b & -c & 0 & 0 \\ -d & -e & 0 & 0 \end{bmatrix}$$

(a) (*10 points.*) Think of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as a block matrix consisting of four 2×2 matrices A, B, C and D. Find the determinant of each of the matrices A, B, C and D.

$$\det(A) = \begin{vmatrix} 0 & a \\ -a & a \end{vmatrix} = 0 \cdot a - (-a)(a) = a^2$$

$$\det(B) = \begin{vmatrix} b & d \\ c & e \end{vmatrix} = be - dc$$

$$\det(C) = \begin{vmatrix} -b & -c \\ -d & -e \end{vmatrix} = \det(-B) = (-1)^2(be - dc) = (be - dc)$$

$$\det(D) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

(b) (*10 points.*) Show that the determinant of $M = (be - dc)^2$ using the Laplace Expansion Formula (pick the row or column you expand about carefully!)

$$\begin{vmatrix} 0 & a & b & d \\ -a & a & c & e \\ -b & -c & 0 & 0 \\ -d & -e & 0 & 0 \end{vmatrix} = d(-1)^{1+4} \begin{vmatrix} -a & a & c \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix} + e(-1)^{4+2} \begin{vmatrix} 0 & a & b \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix}$$

$$= -d \cdot (-1)^{1+3} c \begin{vmatrix} -b & -c \\ -d & -e \end{vmatrix} + e \cdot b(-1)^{1+3} \begin{vmatrix} -b & -c \\ -d & -e \end{vmatrix}$$

$$= -dc(be - dc) + be(be - dc) = (be - dc)(be - dc)$$

$$= (be - dc)^2$$

(c) (*5 points.*) Write down a relationship between the determinant of the block matrix M, i.e. $\det(M)$ and the determinant of each of the blocks, i.e. $\det(A)$, $\det(B)$, $\det(C)$, and $\det(D)$.

$$\det(M) = -[\det(A)\det(D) - \det(B)\det(C)]$$

$$= -[a^2 \cdot 0 - (be - dc)(be - dc)]$$

$$= (be - dc)^2$$

6. *[25 points total.]* **Eigenvalues, Eigenvectors, Diagonalization.**
Let A be an unknown 2×2 matrix with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ corresponding to eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ respectively.

a. *(6 points).* Compute the determinant and trace of the unknown matrix A from the information you currently have.

$$\det(A) = \lambda_1 \lambda_2 = 5 \cdot -1 = -5$$

$$\cancel{\det}\ \text{trace}(A) = \lambda_1 + \lambda_2 = 5 + -1 = 4$$

$$S = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} \frac{1}{-3} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$$

$$SS^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

b. *(10 points).* Since A is similar to a diagonal matrix Λ such that $AS = S\Lambda$, compute A.

$$A = S \Lambda S^{-1}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\frac{1}{3}$$

$$= \begin{pmatrix} 5 & -2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\frac{1}{3}$$

$$= \begin{pmatrix} 3 & 12 \\ 6 & 9 \end{pmatrix}\frac{1}{3} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

c. *(9 points).* Confirm your answers to part (a) and verify that the $\vec{v}_1$ and $\vec{v}_2$ are indeed eigenvectors of A with corresponding eigenvalues λ_1 and λ_2.

$$\text{trace}(A) = 1 + 3 = 4 \checkmark$$

$$\det(A) = 1 \cdot 3 - 4 \cdot 2 = 3 - 8 = -5 \checkmark$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{matrix} 5 \\ 5 \end{matrix} = \overset{\checkmark}{5}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \overset{\checkmark}{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

7. *[25 points total.]* **Projections, Orthogonal Complements, Analytic Geometry, Orthogonal Decomposition.**

Consider $\mathcal{W} = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

a. *(6 points).* Show that $\mathcal{W}$ is the plane $-x + y + z = 0$.

Find vector orthogonal to $\text{span}\left\{\begin{pmatrix}1\\3\\-2\end{pmatrix}, \begin{pmatrix}5\\1\\4\end{pmatrix}\right\}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 5 & 1 & 4 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix}$$

$$\vec{n} = \hat{i}\,14 - \hat{j}\,14 + \hat{k}(-14)$$

Since it goes through the origin $(0,0,0)$ is on plane

$$(1,-1,-1)(x,y,z) = 0$$
$$x - y - z = 0$$
$$-x + y + z = 0$$

b. *(6 points).* Find an orthogonal basis for $\mathcal{W}$.

$$\text{Span}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = W^{\perp}$$

Note $\begin{pmatrix}1\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}5\\1\\4\end{pmatrix} = 5 + 3 - 8 = 0$ so the given basis for W is orthogonal

c. *(6 points).* Find the projection of $\vec{v}$ onto $\mathcal{W}^{\perp}$.

$$\text{proj}_{\begin{pmatrix}1\\-1\\-1\end{pmatrix}} \vec{v} = \frac{\begin{pmatrix}1\\3\\5\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}}{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \frac{1-3-5}{1+1+1}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = -\frac{\cancel{8}}{3}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

d. *(7 points).* Find the orthogonal decomposition of $\vec{v}$ with respect to $\mathcal{W}$ and $\mathcal{W}^{\perp}$, i.e. $\vec{v} = c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3$ where $\mathcal{W} = \text{span}(\vec{w}_1, \vec{w}_2)$ and $\mathcal{W}^{\perp} = \text{span}(\vec{w}_3)$.

$$c_3 = -\frac{8}{3}$$

$$\text{proj}_{\begin{pmatrix}1\\3\\-2\end{pmatrix}}\begin{pmatrix}1\\3\\5\end{pmatrix} = \frac{\begin{pmatrix}1\\3\\5\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}}{\begin{pmatrix}1\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}}\begin{pmatrix}1\\3\\-2\end{pmatrix} = \frac{10-10}{1+9+4}\begin{pmatrix}1\\3\\-2\end{pmatrix} = 0\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$\text{proj}_{\begin{pmatrix}5\\1\\4\end{pmatrix}}\begin{pmatrix}1\\3\\5\end{pmatrix} = \frac{\begin{pmatrix}1\\3\\5\end{pmatrix}\cdot\begin{pmatrix}5\\1\\4\end{pmatrix}}{\begin{pmatrix}5\\1\\4\end{pmatrix}\begin{pmatrix}5\\1\\4\end{pmatrix}}\begin{pmatrix}5\\1\\4\end{pmatrix} = \frac{5+3+20}{25+1+16}\begin{pmatrix}5\\1\\4\end{pmatrix} = \frac{28}{42}\begin{pmatrix}5\\1\\4\end{pmatrix} = \frac{2}{3}\begin{pmatrix}5\\1\\4\end{pmatrix}$$

$$\begin{pmatrix}1\\3\\5\end{pmatrix} = 0\begin{pmatrix}1\\3\\-2\end{pmatrix} - \frac{\cancel{8}}{3}\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}5\\1\\4\end{pmatrix} = \begin{matrix}-7/3 + 10/3\\ 7/3 + 2/3\\ 7/3 + 8/3\end{matrix} = \begin{pmatrix}3/3\\9/3\\15/3\end{pmatrix} = \begin{pmatrix}1\\3\\5\end{pmatrix} ✓$$

8. *[25 points total.]* **Solutions of Linear Systems, Invertibility, Singular.**
Consider the linear system with unknown parameter d

$$\begin{aligned} 1x + 1y + 1z &= 1 \\ 1x + (d+1)y + 3z &= 5 \\ 0x + 2y + dz &= -4 \end{aligned}$$

a. *(10 points).* Show that after applying elementary row operations to the linear system $A\vec{x} = \vec{b}$ the augmented coefficient matrix becomes $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & d & 2 & 4 \\ 0 & 0 & d - \frac{4}{d} & -4 - \frac{8}{d} \end{array}\right]$ (Assume $d \neq 0$)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & d+1 & 3 & 5 \\ 0 & 2 & d & -4 \end{array}\right) \xrightarrow{R_2' = R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & d & 2 & 4 \\ 0 & 2 & d & -4 \end{array}\right) \xrightarrow[R_3' = R_3 - \frac{2}{d}R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & d & 2 & 4 \\ 0 & 0 & d - \frac{4}{d} & -4 - \frac{8}{d} \end{array}\right)$$

$$\left(d - \frac{4}{d}\right) z = \left(-4 - \frac{8}{d}\right)$$

$$\left(\frac{d^2 - 4}{d}\right) z = -4\left(\frac{d+2}{d}\right)$$

$$z = -4 \frac{d+2}{d^2-4}$$

$$\boxed{z = \frac{-4}{d-2}} \qquad \boxed{y = \frac{2}{d}\left(\frac{d+2}{d-2}\right)}$$

$$dy + 2\left(\frac{-4}{d-2}\right) = 2 \Rightarrow dy = 2 + \frac{8}{d-2}$$

$$dy = \frac{2d+4}{d-2} = 2\left(\frac{d+2}{d-2}\right)$$

b. *(5 points).* For what values of d will the linear system have an **infinite** number of solutions?

If $d - \frac{4}{d} = 0$ and $-4 - \frac{8}{d} = 0$

$d^2 - 4 = 0$ and $-4d - 8 = 0$

$d = \pm 2$ and $-4(d+2) = 0 \Rightarrow d = -2$

When $d = -2$ the system will have an infinite # of solutions

c. *(5 points).* For what values of d will the linear system have a **zero** number of solutions?

$$d - \frac{4}{d} = 0 \qquad \text{and} \qquad -4 - \frac{8}{d} \neq 0$$

$$d = \pm 2 \qquad\qquad d \neq -2$$

There will be zero solutions

When $d = 2$ there will be no solutions

d. *(5 points).* For what values of d will the linear system have a **single** solution?

For all other values of $d \neq 2$ or -2

There will be one solution

$$\det(A) = 1 \cdot d \cdot \left(d - \frac{4}{d}\right) \neq 0$$

$$= 1 \cdot d \, \frac{d^2 - 4}{d} \neq 0$$

$$= d^2 - 4 \neq 0$$

When $d \neq \pm 2$, $\det A \neq 0$

Note, when $d = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 0 & 2 & 0 & -4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 0 & 1 & 0 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & -2 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right)$$