# Test 2: Linear Systems

Math 214 Spring 2007 | Friday April 20
©Prof. Ron Buckmire | 2:30pm-3:25pm

Name: Key

**Directions**: Read *all* problems first before answering any of them. There are 7 pages in this test (including this page). This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

You may not discuss the questions on this test with any other student.

**Pledge:** I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

| No. | Score | Maximum |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 20 |
| BONUS | | 10 |
| **Total** | | **100** |

1. **Column Space, Linear Combinations, Solvability.** (*20 points.*)

(a) Give the definition of the phrase "the column space of a matrix."

The column space of a matrix A, usually denoted col(A), is the subspace spanned by the columns of A, i.e. the set of all possible linear combinations of A.

(b) Prove or give a counter-example: IF $A$ is an $m \times n$ matrix and $\vec{b}$ is a vector in $\mathbb{R}^m$ such that the equation $A\vec{x} = \vec{b}$ has one or more solutions, THEN $\vec{b}$ is a linear combination of the columns of $A$.

Statement is TRUE.

$A\vec{x} = \vec{b}$ means there exist an $m \times 1$ vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x}$ is the same as $\vec{b}$.

$$A\vec{x} = \begin{pmatrix} col_1 & col_2 & \cdots & col_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} col_1 \end{pmatrix} + x_2 \begin{pmatrix} col_2 \end{pmatrix} + \cdots + x_n \begin{pmatrix} col_n \end{pmatrix}$$

$A\vec{x}$ <u>is</u> a linear combination of the columns of A and if $A\vec{x} = \vec{b}$ then $\vec{b}$ is equal to a linear combination of the columns of A, i.e. $\vec{b} \in col(A)$

$$\vec{b} \in col(A) \iff A\vec{x} = \vec{b} \text{ has at least 1 solution}$$

2. **Basis, Vector Space.** (*20 points.*)

(a) Give the definition of the phrase "a basis for a vector space."

A basis for a vector space is a collection of vectors that are linearly independent that span the vector space. The number of vectors in a basis for a space is the dimension of that space.

(b) Prove or give a counter-example: IF $\vec{u}$, $\vec{v}$ and $\vec{w}$ are unit vectors in $\mathbb{R}^3$ such that none of them is a multiple of another, THEN $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.

Statement is FALSE.

$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

None of $\vec{u}, \vec{v}$ or $\vec{w}$ is a multiple of the other, but $\vec{u} + \vec{v} = \vec{w}$ so $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\} \neq \mathbb{R}^3$

3. **Eigenvector, Eigenvalue.** (*20 points.*)

(a) Give the definition of the phrase "an eigenvector of a matrix."

An eigenvector of a matrix is a non-zero vector that satisfies the equation $A\vec{x} = \lambda\vec{x}$ where $\lambda$ is known as an eigenvalue and satisfies the polynomial $p(\lambda) = \det(A - \lambda I) = 0$ and A is a square matrix.

(b) Prove or give a counter-example: IF $\vec{v}$ and $\vec{w}$ are eigenvectors of a matrix $A$ such that $\vec{v}$ and $\vec{w}$ have the same eigenvalue $\lambda$, THEN $\vec{v} + \vec{w}$ is an eigenvector of $A$.

Statement is TRUE.

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{w} = \lambda\vec{w}$$
$$\Rightarrow A\vec{v} + A\vec{w} = \lambda\vec{v} + \lambda\vec{w}$$
$$A(\vec{v} + \vec{w}) = \lambda(\vec{v} + \vec{w})$$

Thus $\vec{v} + \vec{w}$, is an eigenvector of A with eigenvalue $\lambda$ if $\vec{v}$ and $\vec{w}$ are also eigenvectors with the same eigenvalue.

4. **Subspace, Dimensional, Orthogonal Complements.** (*20 points.*)

(a) Give the definition of the phrase "a subspace of a vector space."

A subspace of a vector space is a subset of a vector space that
(i) contains the zero vector, i.e $\vec{0} \in S$
(ii) is closed under vector addition
$\vec{v}, \vec{w} \in S \Rightarrow \vec{v} + \vec{w} \in S$

(iii) is closed under scalar multiplication
$\vec{v} \in S, c \in \mathbb{R} \Rightarrow c\vec{v} \in S$

(b) Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ be linearly independent vectors in $\mathbb{R}^3$ and let $\mathcal{W}$ be the orthogonal complement of $\text{span}(\vec{v}, \vec{w})$. Is $\mathcal{W}$ a subspace of $\mathbb{R}^3$? If it is a subspace of $\mathbb{R}^3$, give the dimension of $\mathcal{W}$ and explain how you find its dimension. If $\mathcal{W}$ is not a subspace of $\mathbb{R}^3$, explain why it is not.

$W$ is a subspace of $\mathbb{R}^3$ it is the line through origin orthogonal to the plane represented by $\text{span}(\vec{v}, \vec{w})$.
$W = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$. The dimension of $W$ is 1 since the sum of $\dim W$ and $\dim W^\perp$ must equal 3 and you know $\dim W^\perp = 2$ since its basis contains 2 vectors $\vec{v}$ and $\vec{w}$. A span of a collection of vectors is always a subspace.

5. **Linearly Independence, Invertibility.** (*20 points.*)

(a) Give the definition of the phrase "a linearly independent set of vectors in $\mathbb{R}^n$." (You CAN NOT just say "Not Linearly Dependent!")

A linearly independent set of vectors is a set of $n$ vectors $\{\vec{v}_i\}_{i=1}^{n}$ such that the only solution to $\sum_{i=1}^{n} c_i \vec{v}_i = 0$ is the trivial one $c_i = 0$ for $i = 1$ to $n$.

In other words no vector in a linearly independent set can be written as a linear combination of the remaining vectors in the set.

(b) IF an $n \times n$ matrix $A$ has $n$ linearly independent columns, THEN (Put a CHECK-MARK $\surd$ in the box next to each of the statements below that is true):

- ☑ The rows of $A$ are linearly independent
- ☑ The column space of A is $\mathbb{R}^n$
- ☑ Every row of $A$ is a linear combination of the columns of A
- ☑ The reduced row echelon form of $A$ is the identity matrix
- ☐ $\det(A) = 0$
- ☑ For every vector $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution
- ☑ The null space of $A$ contains no vectors other than the $\vec{0}$ vector
- ☑ $A$ is a non-singular matrix
- ☑ $\text{rank}(A) = n$
- ☑ $A^T$ is invertible.

No explanation needs to be given for why you think your chosen statements are true.

$\det(A) \neq 0$ since $A^{-1}$ exists

**BONUS QUESTION. Linearly Independence, Invertibility.** (*10 points.*)
Provide detailed written explanations for each of the statements in Question 5, explaining why the statement is either TRUE or FALSE.

1. If $n$ columns are linearly independent then $\dim(\mathrm{col}(A)) = n = \text{rank} = \dim(\mathrm{row}(A))$ so there will be $n$ lin independent rows also

2. $\dim \mathrm{col}(A) = n$ only $n$ dimensional subspace of $\mathbb{R}^n$ IS $\mathbb{R}^n$

3. $\mathrm{col}(A) = \mathbb{R}^n$, every row of $A$ is in $\mathbb{R}^n$ so every row is a linear comb of the columns

4. Row reduction on $A$ with $n$ lin independent columns and rows must equal $I$ since rank $= n$

5. $\det(A) \neq 0$ since $\text{rref}(A) = I$

6. $\det(A) \neq 0 \Rightarrow A^{-1}$ exists $\Rightarrow$ ~~$A^{-1}$ ...~~ $\vec{x} = A^{-1}\vec{b}$ is a unique soln to $A\vec{x} = \vec{b}$

7. Since $A^{-1}$ exists $A\vec{x} = \vec{0} \Rightarrow \vec{x} = A^{-1}\vec{0} = \vec{0}$ is the only solution.
Could also use Rank theorem rank + nullity $= n$
$n$ + nullity $= n$
nullity $= 0$
~~dim col A + dim null~~
$\dim(\mathrm{null}(A)) = 0 \Rightarrow \mathrm{null}(A) = \{\vec{0}\}$

8. $\det(A) \neq 0 \Leftarrow A$ is non-singular or invertible

9. $\text{rank}(A) = n = \dim(\mathrm{col}(A)) = \#$ of lin ind columns

10. $(A^{-1})^T = (A^T)^{-1}$ and $A^{-1}$ exists so $(A^T)^{-1}$ exists.