Test 1: LINEAR SYSTEMS

Math 214 Spring 2007
©Prof. Ron Buckmire

Friday March 2
2:30pm-3:25pm

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Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, __________________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		25
3		30
4		25
BONUS		10
Total		**100**

1. Span, Linear Independence, Rank. *20 points.*

Consider the matrix $A = \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix} \rightarrow \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(a) (*4 points.*) Find the reduced row echelon form of A, rref(A).

$\text{rref}(A) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(b) (*4 points.*) With or without your knowledge of rref(A), what is the rank of the matrix A? **EXPLAIN YOUR ANSWER.**

rank $= 1$

Number of non-zero rows

(c) (*4 points.*) With or without your knowledge of rref(A), what is the span of the columns of matrix A? **EXPLAIN YOUR ANSWER.**

$\text{span}\left\{\begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 6 \\ 18 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} =$ the line $y = 3x$

(d) (*4 points.*) With or without your knowledge of rref(A), discuss the linear independence of the columns of the matrix A. **EXPLAIN YOUR ANSWER.**

$\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 \\ 18 \end{pmatrix}$ The colums are linear dependent

(e) (*4 points.*) With or without your knowledge of rref(A), discuss whether the matrix A^{-1} exists. **EXPLAIN YOUR ANSWER.**

A^{-1} does not exist since $\text{rref}(A) \neq I$

2. Row reduction, Reduced Row Echelon Form, Identity, Invertibility. *25 points.*

(a) (*8 points.*) Show that $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & 2 & 6 \end{bmatrix}$ can be transformed into $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ using elementary row operations.

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (*8 points.*) Show that I_3 can be transformed into $B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ using elementary row operations.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) (*9 points.*) Is it possible to transform A into B using elementary row operations? Explain why you can not or explain why you can, without going through the actual work of proving the result. Is A invertible? Is B invertible? How do you know? EXPLAIN YOUR ANSWERS.

$$A \to I_3 \to B$$

Yes it's possible.

$$(E_k \cdots E_1) A = I_3$$
$$A = (E_k \cdots E_1)^{-1}$$
$$A^{-1} = (E_k \cdots E_1)$$

$$EI = B \qquad E^{-1} = B^{-1}$$
$$I = E^{-1} B$$

3. Matrix Operations, Trace, Transpose. *30 points.*

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

The **trace** of a $n \times n$ matrix A (where A_{ij} is the element in the i^{th} row and j^{th} column) is defined as **the sum of the diagonal elements of A** and denoted $\text{tr}(A)$. In other words, $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$

(a) *10 points.* **TRUE or FALSE?** "The trace of the $n \times n$ identity matrix I_n, $\text{tr}(I_n)$ equals n."

TRUE

$tr(I_n) = \sum_{i=1}^{n} 1 = n$

$I_n = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n}$

(b) *10 points.* **TRUE or FALSE?** "$\text{tr}(A)\text{tr}(B) = \text{tr}(AB)$ for every $n \times n$ matrix A and B."

FALSE

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

$tr A = 5$

$tr B = 13$

$tr A \cdot B = 65$

$tr(AB) = 64$

(d) *10 points.* **TRUE or FALSE?** "$\text{tr}(AA^T)$ equals the sum of the square of each element in A for every $n \times n$ matrix A."

TRUE

$(AA^T)_{ii} = row_i(A) \cdot col_i(A^T)$

$= row_i(A) \cdot row_i(A) = \sum_{i=1}^{n} a_{ii}^2$

$= |row_i(A)|^2$

$tr(AA^T) = \sum_{i=1}^{n} (AA^T)_{ii} = \sum_{i=1}^{n} |row_i(A)|^2 = \sum_{i=1}^{n} A_{i1}^2 + A_{i2}^2 + \dots A_{in}^2$

4. Linear Systems, Equations of Planes and Lines. *23 points.*

Consider the linear system

$$\begin{aligned} 3x + 5y - 4z &= 0 \\ -3x - 2y + 4z &= 0 \\ 6x + y - 8z &= 0 \end{aligned}$$

(a) (*10 points.*) Find the non-trivial solution(s) of the linear system.

$$\begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \to \begin{pmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$3x - 4z = 0 \Rightarrow x = \frac{4}{3}z$

$y = 0$

$\vec{x} = t\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ for any $t \in \mathbb{R}$

(b) (*10 points.*) What is the geometrical object which represents the solution of the linear system? Write down its equation in vector form.

The object is a line, The intersection of 3 planes through the origin.

In vector form the equation of the line is

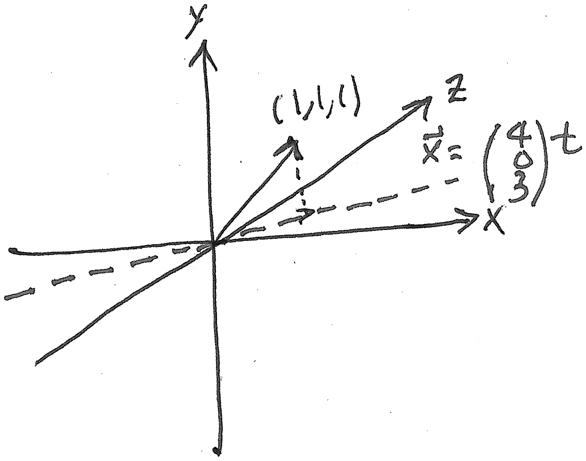
$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

(c) (*5 points.*) What is the smallest distance between the geometrical object which represents the solution of the linear system and the origin $(0, 0, 0)$. EXPLAIN YOUR ANSWER.

Since the line goes through the origin, the distance is zero.

BONUS QUESTION. Analytic Geometry, Projections. (*10 points.*)
Find the smallest distance between the geometrical object which represents the solution to the linear system in Question 4 and the point $(1, 1, 1)$.

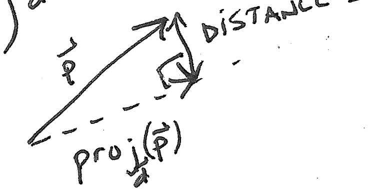
The point $(0,0,0)$ is on the line.



We need to find projection of $\vec{p} = \cancel{(1,1,1)} \begin{pmatrix}1\\1\\1\end{pmatrix}$ in the $\vec{d}$ direction $\begin{pmatrix}4\\0\\3\end{pmatrix}$

DISTANCE + $\text{proj}_{\vec d}(\vec p) = \vec p$

DISTANCE $\vec D = \vec p - \text{proj}_{\vec d}(\vec p) = \vec p - \left(\frac{\vec p \cdot \vec d}{\vec d \cdot \vec d}\right)\vec d$



$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}}{\begin{pmatrix}4\\0\\3\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}} \begin{pmatrix}4\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{7}{25}\begin{pmatrix}4\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}28/25\\0\\21/25\end{pmatrix} = \begin{pmatrix}-3/25\\1\\4/25\end{pmatrix}$$

$\frac{1}{25}\sqrt{650} = \frac{\sqrt{26\cdot 25}}{25} = \frac{5}{25}\sqrt{26}$

$$\boxed{\frac{\sqrt{26}}{5} = |\vec D|} = \sqrt{\left(\frac{3}{25}\right)^2 + \left(\frac{4}{25}\right)^2 + 1^2} = \frac{1}{25}\sqrt{3^2+4^2+25^2}$$

The line is within $\sqrt{26}/5$th unit of the point $(1,1,1)$ at the closest point.

Confirm $\vec D$ is orthogonal to $\vec p$ so $\vec D$ is minimum distance

$$\vec D \cdot \text{proj}_{\vec d}(\vec p) = \left(\vec p - \left(\frac{\vec p\cdot\vec d}{\vec d\cdot\vec d}\right)\vec d\right)\cdot\left(\frac{\vec p\cdot\vec d}{\vec d\cdot\vec d}\right)\vec d$$

$$= \frac{(\vec p\cdot\vec d)^2}{\vec d\cdot\vec d} - (\vec p\cdot\vec d)\frac{(\vec p\cdot\vec d)}{(\vec d\cdot\vec d)^2}\vec d\cdot\vec d = \frac{(\vec p\cdot\vec d)^2}{\vec d\cdot\vec d} - \frac{(\vec p\cdot\vec d)^2}{\vec d\cdot\vec d} = 0$$