Linear Systems

Math 214 Spring 2006 | Fowler 307 MWF 2:30pm - 3:25pm

 | http://faculty.oxy.edu/ron/math/214/06/

Class 26: **Friday April 7**

TITLE Orthogonality and Projections Revisited
CURRENT READING Poole 5.1

Summary
We shall return to the investigation of projections and orthogonality, this time with more increased generality.

Homework Assignment
Poole, Section 5.1: 3,4,5,6,8, **9,12**, *13,16,* **17,30,31**. *EXTRA CREDIT 28, 33.*

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1. Orthogonal Bases

DEFINITION

An **orthogonal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that is an **orthogonal set** of vectors. An orthogonal set of vectors is a collection of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ where *every* pair of distinct vectors is orthogonal to each other, i.e. $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$.

Theorem 5.1

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ then those vectors are linearly independent.

EXAMPLE

Show that $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$.

Theorem 5.2

Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in $\mathcal{W}$. THEN the unique scalars $c_1, c_2, c_3, \ldots, c_n$ (also known as coordinates) where $\vec{w} = \sum_{i=1}^{n} c_i \vec{v}_i$ are given by

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

EXAMPLE

Let's show how this formula for the coordinates is derived. (Doesn't it look familiar??)

Exercise

Given the orthogonal basis $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the vector $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinates of $\vec{w}$ with respect to β, i.e. $[\vec{w}]_\beta$.

DEFINITION

An **orthonormal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that consists of an **orthonormal set** of vectors. An orthonormal set of vectors is a collection of orthogonal unit vectors $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots, \vec{q}_k\}$ where $\vec{v}_i \cdot \vec{v}_j = \delta_{i,j}$. The symbol $\delta_{i,j}$ is known as the Kronecker delta function and has the property that $\delta_{i,j} = 0$ when $i \neq j$ and $\delta_{i,j} = 1$ when $i = j$.

Exercise

Form an orthonormal basis for $\mathbb{R}^3$ from the orthogonal basis β given in the previous **Exercise**.

2. Orthogonal Matrices

DEFINITION

A $n \times n$ matrix Q is said to be an **orthogonal matrix** if the columns (and rows) of the matrix form an orthonormal set.

Theorem 5.4

The columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

Theorem 5.5

A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Theorem 5.8

Let Q be an orthogonal matrix.

(a) Q^{-1} is orthogonal.

(b) $\det(Q) = \pm 1$.

(c) If λ is an eigenvalue of Q, then $|\lambda| = 1$.

(d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.

EXAMPLE

Let's form a square orthogonal matrix from the orthonormal basis found in the previous exercise and illustrate some of the results from Theorem 5.4, 5.5 and 5.8.