Linear Systems

Math 214 Spring 2006 — Fowler 307 MWF 2:30pm - 3:25pm

 — http://faculty.oxy.edu/ron/math/214/06/

Class 22: **Monday March 27**

TITLE Determinants
CURRENT READING Poole 4.3

Summary
Let's explore the wonderful world of *eigenvectors*, *eigenvalues* and *eigenspaces* of a square $n \times n$ matrix.

Homework Assignment
*Poole, Section 4.3: 4,***5,10***,15,16,***17,18***,20,21,***23,33***. EXTRA CREDIT 34,36,38.*

DEFINITION

The eigenvalues of a square $n \times n$ matrix A satisfy the **characteristic polynomial** of the matrix A, given by $\det(A - \lambda I) = 0$.

EXAMPLE

Find the eigenvalues and corresponding eigenspaces of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

DEFINITION

The **algebraic multiplicity** of an eigenvalue is the multiplicity of this eigenvalue as a root of the characteristic polynomial. The **geometric muliplicity** of an eigenvalue λ is the *dimension* of the corresponding eigenspace E_λ, i.e. the number of vectors in a basis for the eigenspace.

Exercise

Write down the algebraic and geometric multiplicity of the eigenvalues of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

$\boxed{\textbf{Theorem 4.15}}$

The eigenvalues of a triangular matrix (lower triangular, upper triangular or diagonal) are simply the entries along its main diagonal.

$\boxed{\textbf{Theorem 4.16}}$

Let A be a square matrix with eigenvalue λ and eigenvector $\vec{x}$
(i) For any integer n, λ^n is an eigenvalue of A^n with correspnding eigenvector $\vec{x}$
(ii) If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$

$\boxed{\textbf{Theorem 4.18}}$

A square matrix A is invertible if and only if 0 is NOT an eigenvalue of A.

$\boxed{\text{EXAMPLE}}$

Poole, page 296, #19. (a) Show that for any square matrix A, A^T and A have the same characteristic polynomial and thus the same eigenvalues.

(b) Give an example of a 2x2 matrix A for which A^T and A have different eigenspaces.

$\boxed{\textbf{Exercise}}$

Show that the eigenvalues $A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$ are 5 and -2 and $E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$ and $E_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Find the eigenvalues of $3A$, A^{-1}, A^2 and $A + I$

Linear Independence of Eigenvectors

Theorem 4.19

Suppose the $n{\times}n$ matrix A has m eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. IF $\vec{x}$ is a vector in $\mathbb{R}^n$ that can be written as a linear combination of these vectors, THEN

$$A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + c_3\lambda_3^k\vec{v}_3 + \ldots c_m\lambda_m^k\vec{v}_m$$

EXAMPLE

Let's use this result to show that $\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}^6 \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 46747 \\ 47195 \end{bmatrix}$

Theorem 4.20

Let A be an $n \times n$ matrix with m distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$. Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ are linearly independent.

Properties of the Eigenvalues of a $n \times n$ Matrix

The **Product** of the eigenvalues equals the **determinant** of the $n \times n$ matrix.

$$\lambda_1\lambda_2\lambda_3\ldots\lambda_n = |A|$$

The **Sum** of the eigenvalues equals the **trace** of the $n \times n$ matrix (the sum of the diagonal entries)

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} A_{ii}$$