Class 13: **Wednesday February 22**

SUMMARY Elementary Matrices and the Fundamental Theorem of Inverses
CURRENT READING Poole 3.3

Summary
Previously we have learned how to use elimination to convert a system of linear equations into an upper-tridiagonal system which we can use back-substitution on to find the solution. Today we want to see how the steps we took to do the elimination (i.e. identifying the pivots and multipliers and using them to eliminate specific coefficients) can be achieved in an identical fashion through matrix multiplication.

Homework Assignment
*HW # 13: Section 3.3 # 24,25,31,32,***37,38***,40,42,43,***45,46,51,54***. EXTRA CREDIT 65, 71.*

Consider the linear system

$$\begin{aligned} 3x + 2y + 1z &= 1 \\ 6x + 4y - 3z &= 3 \\ 9x + 7y + 1z &= 0 \end{aligned} \tag{1}$$

1. Matrix Multipication as an Elimination Technique

Compute

$$P \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & -3 \\ 9 & 7 & 1 \end{bmatrix} =$$

Compare the product to the original matrix A. Compare P to I. Do you notice a trend?

Recall the linear system above in (1). What happens to the coefficient matrix after the first step in the elimination process?

[i.e. what does the row form and/or the matrix look like now?]

PIVOT: 3

MULTIPLIER: -6/3

THE AUGMENTED MATRIX A' becomes: $\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right]$

Consider the matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ -6/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Compute $E \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ -6/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & -3 \\ 9 & 7 & 1 \end{bmatrix} =$

We rename E as $E_{2,1}$ (or E_{21}) to reflect which element of the coefficient matrix has been eliminated.

The linear system thus moves from $A\vec{x} = \vec{b}$ to $E_{2,1} \cdot A\vec{x} = E_{2,1} \cdot \vec{b}$

What is the next step in the elimination process? We want to eliminate element ____________ from the coefficient matrix. The PIVOT is ____________ and the multiplier is ____________. Therefore the E matrix to accomplish this elimination will be:

The result will be $\begin{bmatrix} 3 & 2 & 1 & | & 1 \\ 0 & 0 & -5 & | & 1 \\ 0 & 1 & -2 & | & -3 \end{bmatrix}$

Q: How do we get this into upper-triangular form? (Remember PIVOTS **can not** be zero!)
A: Do a row exchange! Swap rows 2 and 3!

This is equivalent to multiplying by $P_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

The final form of all the matrix multiplication to do elimination for this system is:

$$P_{32} \cdot E_{31} \cdot E_{21} \cdot A \cdot \vec{x} = P_{32} \cdot E_{31} \cdot E_{21} \cdot \vec{b}$$

So, the final form of the augmented matrix is: $\begin{bmatrix} 3 & 2 & 1 & | & 1 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & -5 & | & 1 \end{bmatrix}$

which in row form is

Back substitute to obtain the solution to the system.

(The answers should be $x = 8/3, y = -17/5, z = -1/5$.)

$\boxed{\text{EXAMPLE}}$ Use matrix multiplication to solve the following linear system.

$$\begin{aligned} 1x + 2y + 3z &= 9 \\ 2x - 1y + 1z &= 8 \\ 3x + 0y - 1z &= 3 \end{aligned}$$

1. Write down the augmented coefficient matrix A' (3x4 matrix)

2. You should be able to write down E_{21} and E_{31} right now.

You can not write down the final matrix to use because the pivot may have changed through the actions of these first operations.

3. Compute $E_{31} \cdot E_{21} \cdot A'$ (Your result should be a __________ matrix).

4. Write down the third matrix you need to multiply by to complete the process.

5. Write down the upper-triangular form of the system and back-solve to find $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which solves the system

$\boxed{\textbf{Theorem 3.12}}$

The Fundamental Theorem of Invertible Matrices (Version 1). Let A be a $m \times n$ matrix. Each of the following statements is equivalent:

(a) A is invertible.

(b) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$.

(c) $A\vec{x} = \vec{0}$ has only the trivial solution.

(d) The reduced row echelon form of A, rref(A), is I_n.

(e) A is a product of elementary matrices

$\boxed{\text{EXAMPLE}}$

Let's look at page 171 of Poole and see if we can follow the proof. The method used is to show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$. Pretty cool, eh?