Linear Systems

Math 214 Spring 2006 — Fowler 307 MWF 2:30pm - 3:25pm

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Class 12: **Friday February 17**

SUMMARY The Inverse Matrix
CURRENT READING Poole 3.3

Summary
We will introduce a very important concept, the Inverse Matrix.

Homework Assignment
HW # 12: Section 3.3 # 2,5, **9,10,19,20,21**, *22,23 EXTRA CREDIT # 13*

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1. Inverse Matrix

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Write down the product of M and A. That is, MA and AM.

We call M, the matrix which when multiplied by A produces the identity matrix. It is denoted A^{-1}.

It has the property that $A^{-1}A = AA^{-1} = I$

The factor $ad - bc$ is known as the **determinant** of the matrix A. We will learn more about how to compute determinants and their significance later. However, it is true that if the determinant of a matrix equals zero, then that matrix is NOT invertible, i.e. det $A = 0 \Rightarrow A^{-1}$ doesn't exist. It is also true that if $A^{-1} doesnotexist \Rightarrow \det(A) = 0$.

Theorem 3.6
If A is an invertible matrix, then its inverse A^{-1} is **unique**.

Theorem 3.7
If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$ for **any** $\vec{b}$ in $\mathbb{R}^n$.

2. Computing Inverses: Gauss-Jordan Elimination

In order to actually generate or find an inverse matrix we use a process called Gauss-Jordan elimination. This is identical to the Gaussian elimination process we already know, except extended.

Consider the system

$$
\begin{aligned}
1x + 2y - 1z &= 1 \\
2x + 2y + 4z &= 3 \\
1x + 3y - 3z &= 0
\end{aligned}
$$

Write down the augmented matrix with the identity matrix as the right hand side.

$$
\left[\begin{array}{ccc|ccc}
1 & 2 & -1 & 1 & 0 & 0 \\
2 & 2 & 4 & 0 & 1 & 0 \\
1 & 3 & -3 & 0 & 0 & 1
\end{array}\right]
$$

We will do Gaussian Elimination on this system until we have produced the identity matrix on the left 3x3 matrix.

3. Properties of Inverses

(1) $(A^{-1})^{-1} = A$

(2) $(AB)^{-1} = B^{-1}A{-1}$

(3) $(ABC)^{-1} = C^{-1}B^{-1}A{-1}$

(4) $(A^{-1})^n = (A^n)^{-1}$ for positive integers n

(5) $(A^{-1})^T = (A^T)^{-1}$

(6) $\dfrac{1}{c}A^{-1} = (cA^{-1}$ for positive scalars c

$\boxed{\textbf{Exercise}}$

Consider $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 \\ 2 & 3 \end{bmatrix}$. Show that $A^{-1}B^{-1} = (BA)^{-1}$.

4. Determining Singularity

If the determinant of a coefficient matrix is zero, then the system is singular (no solution or infinite number of solutions) and thus the linear system can not be solved.

$$\det(A) = 0 \longleftrightarrow A^{-1}\text{doesn't exist}$$

So it is NOT always possible to find A^{-1}. A^{-1} exists ONLY IF a $n \times n$ matrix A has rank(n).

5. Using Gauss-Jordan To Solve Linear Systems

Gauss-Jordan takes the augmented matrix $\left[\begin{array}{c} A|I \end{array}\right]$ and converts it into $\left[\begin{array}{c} I|A^{-1} \end{array}\right]$.

Q: What has happened to each block matrix in the augmented matrix?
A: Each block matrix been multiplied by by A^{-1}.

Therefor Gauss-Jordan can also take the matrix $\left[\begin{array}{c} A|I|\vec{b} \end{array}\right]$ and convert into $\left[\begin{array}{c} I|A^{-1}|A^{-1}\vec{b} \end{array}\right]$

Why is this useful?

Gauss-Jordan works by solving n linear systems at once.
For a 3x3 system it is solving $A\vec{x_1} = \vec{e_1}$, $A\vec{x_2} = \vec{e_2}$ and $A\vec{x_3} = \vec{e_3}$
where $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\vec{x_1}$, $\vec{x_2}$ and $\vec{x_3}$ which solve the 3 equations above are simply the columns of the inverse matrix.

Example
Consider the system (with $d \neq 0$)

$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \\ 1 & (d+1) & 3 & | & 0 & 1 & 0 & 5 \\ 0 & 2 & d & | & 0 & 0 & 1 & -4 \end{array}\right]$ Let's use Gauss-Jordan to find the solution