Test 2: Linear Systems

Math 214 Friday April 21 2006
Ron Buckmire 2:30pm-3:25pm

Name: Key

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test (and this cover page). **There is a theme to this test, i.e. all the questions are related.** This is a one hour, no-notes, closed book, test. No calculators. You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, __________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		20
3		20
4		20
5		20
BONUS		10
Total		**100**

1. (*20 points.*) **Associated Subspaces of a Matrix.**
Write down the dimension and a basis for **EACH** of the four fundamental subspaces of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A^T) \qquad \mathrm{rank} = 2$$

$$\dim \mathrm{null}(A) = 1$$

$$\mathrm{null}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim \mathrm{col}(A) = 2$$

$$\mathrm{col}(A) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$\mathrm{row}(A) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} = \mathrm{col}(A) \qquad \dim \mathrm{row}(A) = 2$$

$$\dim(\mathrm{null}(A^T)) = 1$$

$$\mathrm{null}(A^T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. (*20 points.*) **Eigenvalues, Eigenvectors, Eigenspaces.**
Write down the dimension and a basis for **EACH** of the eigenspaces of
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda)$$

$$= -\lambda^3 + \lambda + \lambda$$

$$0 = -\lambda^7 + 2\lambda$$

$$= \lambda(2 - \lambda^2)$$

$$\lambda = 0 \text{ or } 2 = \lambda^2 \Rightarrow \lambda = \pm\sqrt{2}$$

$$E_0 = \text{null}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} \quad (\text{from Question } 1)$$

$$E_{\sqrt{2}} = \text{null}(A - \sqrt{2}I) = \text{null}\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} = \text{null}\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix}$$

$$= \text{null}\begin{pmatrix} -\sqrt{2} & 0 & -\sqrt{2} \\ 1 & 0 & 1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} = \text{null}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$E_{-\sqrt{2}} = \left(\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & -1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} \sqrt{2} & 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & +1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\}$$

3. (*20 points.*) **Definition of a Subspace.**

(a) (*9 points*) Prove that the vector space $\mathcal{V}$ defined below IS a subspace of $\mathbb{R}^3$.

$$\mathcal{V} = \left\{ \vec{v} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_3 = 0, : x_1, x_2, x_3 \in \mathbb{R} \right\} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 \end{pmatrix} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad x_1, x_2 \in \mathbb{R} \right\}$$

$\vec{0} \in V$? Yes! If $x_1 = x_2 = x_3 = 0$
$x_1 + x_3 = 0 \Rightarrow \vec{0} \in \mathcal{V}$

closed under scalar multiplication? Yes!

$c\vec{v} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}$ where $x_1 + x_3 = 0 \Rightarrow cx_1 + cx_3 = c(x_1 + x_3) = 0$

So $c\vec{v} \in \mathcal{V}$

closed under vector addition? Yes!

$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$

but $\vec{v} \in V$ and $\vec{w} \in \mathcal{V} \Rightarrow v_1 + v_3 = 0$ and $w_1 + w_3 = 0$
$\Rightarrow v_1 + v_3 + w_1 + w_3 = (v_1 + w_1) + (v_3 + w_3) = 0$
$\Rightarrow \vec{v} + \vec{w} \in \mathcal{V}$

(b) (*6 points*). Mathematically describe $\mathcal{V}^\perp$, the orthogonal complement to $\mathcal{V}$.

Note: $\mathcal{V} = \text{col}(A)$ from Question 1

So $\mathcal{V}^\perp = \text{null}(A^T)$ from Question 1

$\mathcal{V}^\perp = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \vec{v} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{matrix} x_2 = 0, \\ x_1 = x_3 \end{matrix} ; x_1, x_2, x_3 \in \mathbb{R} \right\}$

(c) (*5 points*). Give the dimensions of $\mathcal{V}$ and $\mathcal{V}^\perp$. What kind of geometric object is each of these subspaces? (Name and describe them.)

$\dim \mathcal{V} = 2$ $\quad$ $\mathcal{V}$ is a plane through the origin in $\mathbb{R}^3$

$\dim \mathcal{V}^\perp = 1$ $\quad$ $\mathcal{V}^\perp$ is a line thru the origin in $\mathbb{R}^3$

4. (*20 points.*) **Orthogonalization, Normalization.**

Consider the set of vectors $\beta = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, which is claimed to be a basis for the previously defined vector space $\mathcal{V}$.

(a) (*10 points*) Is β is a basis for $\mathcal{V}$? **Explain Your Answer.**

Yes β is a basis for $\mathcal{V}$.

$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \in \mathcal{V}$ since $-1+1=0$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{V}$ since $1+-1=0$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ so they are lin ind

Two lin independent elements of a 2-D vector space form a basis for $\mathcal{V}$

(b) (*5 points*) Find an *orthogonal* basis for $\mathcal{V}$.

In Question 1 $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is another basis for $\mathcal{V}$ and these vectors are orthogonal

In Question 2, $\{E_{\sqrt{2}}, E_{-\sqrt{2}}\}$ is another orthogonal basis for $\mathcal{V}$

(c) (*5 points*) Find an *orthonormal* basis for $\mathcal{V}^{\perp}$.

an ~~orthogonal~~ orthonormal basis ~~Span~~ for $\mathcal{V}^{\perp}$ is $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}$ since $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ was a basis for $\mathcal{V}^{\perp}$

5. (*20 points.*) **Orthogonal Decomposition, Projection.**
Putting it all together: The goal of all of our work on the previous 4 problems has been to try to write a given vector in $\mathbb{R}^3$ as a linear combination of its 3 components in three orthogonal directions, split between vector(s) from $\mathcal{V}$ and vector(s) from $\mathcal{V}^\perp$. In other words, we want to write $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathcal{V}$.

(a) (*5 points*). Find the projection of $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto $\mathcal{V}$.

$$\mathrm{proj}_{\mathcal{V}}(\vec{b}) = \vec{b} - \mathrm{proj}_{\mathcal{V}^\perp}(\vec{b}) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal basis for $\mathcal{V}$

$$\mathrm{proj}_{\mathcal{V}}(\vec{b}) = \mathrm{proj}_{\vec{v}_1}(b) + \mathrm{proj}_{\vec{v}_2}(\vec{b})$$

$$= \frac{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ +1/2 \end{pmatrix}$$

(b) (*5 points*). Find the projection of $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto $\mathcal{V}^\perp$.

$$\mathrm{proj}_{\mathcal{V}^\perp}(\vec{b}) = \frac{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(c) (*10 points*). Write down the vector $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ as a linear combination of three vectors, where these three vectors form an orthogonal basis for the union of $\mathcal{V}$ and $\mathcal{V}^\perp$.

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

BONUS (*10 points.*) **Essay question.** Discuss how the fundamental subspaces from Question 1 and the eigenspaces of Question 2 are related to the vector spaces $\mathcal{V}$ and $\mathcal{V}^\perp$ in Question 3. How does these relationships (and other information) assist you in the task to orthogonally decompose the given random vector into three orthogonal directions in Question 5?

The fundamental subspaces in Question 1 split $\mathbb{R}3$ into two orthogonal subspaces, a plane and a line. The 2-D subspace is the row and column space of this symmetric matrix and the 1-D orthogonal complement is the null spaces. ($A = A^T \Rightarrow$ null A = null A^T)

When you find all the eigenspaces of this symmetric matrix A you are finding an orthogonal basis for $\mathbb{R}^3$ consisting of $\{E_0, E_{\sqrt{2}}, E_{-\sqrt{2}}\}$.

Question 3 is a formal way of writing $E_0 = \text{null}(A) = \mathcal{V}^\perp$ and $\mathcal{V} = \text{col}(A) = \text{row}(A) = E_{\sqrt{2}} \cup E_{-\sqrt{2}}$.

Question 4 gives you ~~an~~ a non-orthogonal basis for $\mathcal{V}$ to distinguish "everyday" basis from orthogonal basis.

Question 5 you can use your orthogonal basis for $\mathcal{V}$ from 4 (or 2 or 1!) along with the basis for $\mathcal{V}^\perp$ (which must be orthogonal to your basis for $\mathcal{V}$) to decompose a random vector into 3 orthogonal pieces in $\mathbb{R}^3$.