Test 2: Linear Systems

Math 214 Monday, April 25, 2005

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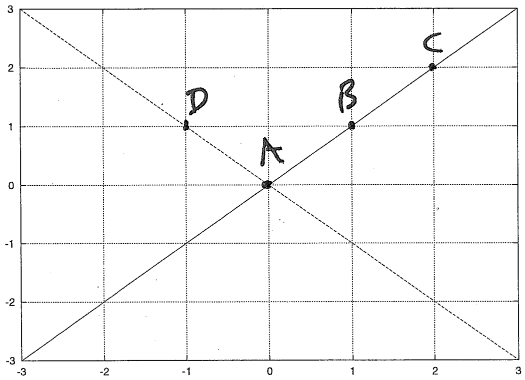
Directions: Read *all* problems first before answering any of them. As usual the questions are all related. **It is probably worth your while to try and spend some time to see the connections between all the questions.**

This is a one hour, open-notes, open book, test.
CALCULATORS ARE NOT ALLOWED.

You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded separate from your "scratch work."

No.	Score	Maximum
1		20
2		30
3		20
4		20
5		10
BONUS		10
Total		**100**

1. (*20 points.*) **Vector Spaces, Orthogonal Complements, Definition of Subspace.** Consider the graph below of the lines $y = x$ and $y = -x$ which represent two vector spaces $\mathcal{V}$ and $\mathcal{W}$ in $\mathbb{R}^2$.



(a) (*10 points*) Write down a definition of one of the vector spaces and prove that it is a valid subspace of $\mathbb{R}^2$.

$\vec{V} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} c, \, c \in \mathbb{R} \right\}$

$\vec{V}_1 + \vec{V}_2 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (c+d)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{V}$

$K\vec{V}_1 = Kc\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (cK)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{V}$

If $c=0$, $\vec{V} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{V}$

$\mathcal{V}$ is closed under vector addition, scaler multiplication and contains $\vec{0}$.

(b) (*10 points*) Show that the other vector space in the picture is the orthogonal complement of the vector space you chose in part (a).

$\vec{V} = \left\{ c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, c \in \mathbb{R} \right\} \in \mathcal{V}$

$\vec{w} = \left\{ d\begin{pmatrix} 1 \\ -1 \end{pmatrix}, d \in \mathbb{R} \right\} \in \mathcal{W}$

$\vec{V} \cdot \vec{w} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot d\begin{pmatrix} 1 \\ -1 \end{pmatrix} = cd \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = cd \cdot 0 = 0$

$\mathcal{V} = \mathcal{W}^{\perp}$

$\mathcal{V}^{\perp} = \mathcal{W}$

2. (*30 points total.*) **General Solution of $A\vec{x} = \vec{b}$**
Consider the points **A** $(0,0)$, **B** $(1,1)$, **C** $(2,2)$ and **D** $(-1,1)$ which are elements in either $\mathcal{V}$ or $\mathcal{W}$ from Question 1. The point of this question is trying to find the equation of a parabola $y = c_0 + c_1 x + c_2 x^2$ which goes through some or all of these points.
For each of the following linear systems, **find the general solution** and thus **explain what the solution represents**: the equation of a parabola through *which* points? How *many* such parabolas exist in each case?

(a) (*10 points*) Consider $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\vec{x} = x_3 \begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Parabola through A and B, ∞ # of sol'ns

(b) (*10 points*) Consider $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

1 solution, $y = x^2$ through A, B and D

$\vec{x}_n = \vec{0}$

$\vec{x}_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) (*10 points*) Consider $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$.

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 4 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

no solution

$\vec{x}_n = \vec{0}$

$\vec{x}_p$ doesn't exist

No parabola exists through all 4 points

3. (*20 points.*) **Associated Subspaces of a Matrix, Basis.**

Write down a **basis** for and the **dimension** of EACH of the subspaces associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{x}_n = \vec{0}$$

$$N(A) = \vec{0} \qquad C(A) = \left\{ x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix} ; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$\dim N(A) = 0 \qquad \dim C(A) = 3$$

$$C(A^T) = \mathbb{R}^3 \qquad \dim C(A^T) = 3$$

$$\dim (C(A) + N(A^T)) = m$$
$$3 + 1 \quad = 4$$

$$\dim C(A^T) + \dim N(A) = n$$
$$3 + 0 \quad = 3$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathrm{rref}(A^T)$$

$$N(A^T) = \left\{ \begin{pmatrix} 3 \\ -3 \\ -1 \\ 1 \end{pmatrix} c, c \in \mathbb{R} \right\} \qquad \begin{matrix} \dim N(A^T) = 1 \\ \dim C(A^T) = 3 \end{matrix}$$

$$C(A^T) = \mathrm{span}\left\{ (1\ 0\ 0), (1\ 1\ 1), (1\ -1\ 1) \right\} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

4. (*20 points.*) **Projection.**

Find the projection of the vector $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ onto the vector space spanned by the vectors

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}$.

$$\vec{p} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} = \begin{matrix} 4/5 \\ 6/5 \\ 2/5 \\ 8/5 \end{matrix}$$

$4\times 2 \quad 2\times 1$

$$\vec{p} = \frac{2}{5}\begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = A\hat{x}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \qquad (A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & 2 & 4 \\ 2 & 6 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & 3 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -5 & -2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 2/5 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & 4/5 \\ 0 & 1 & 2/5 \end{array}\right)$$

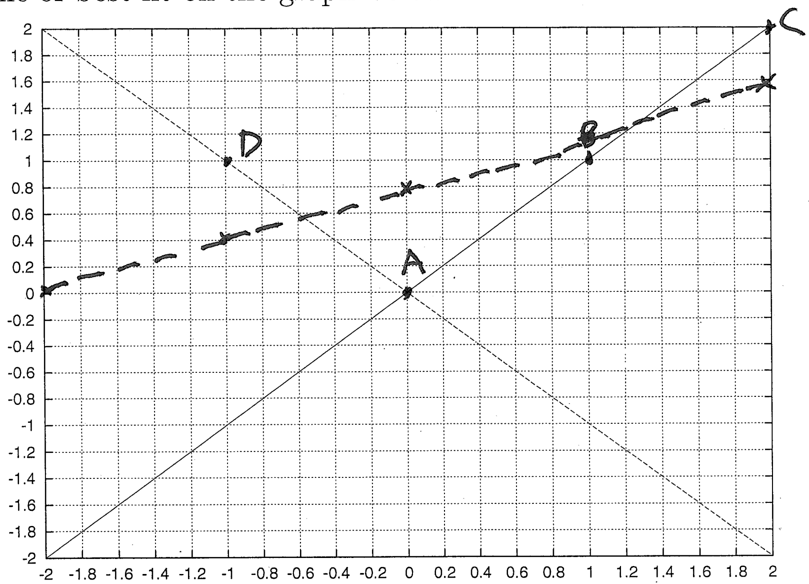
$$A(A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 6 & -2 \\ 4 & 2 \\ 8 & -6 \\ 2 & 6 \end{pmatrix}$$

$$A(A^TA)^{-1}A^T = \frac{1}{20}\begin{pmatrix} 6 & -2 \\ 4 & 2 \\ 8 & -6 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 6 & 4 & 8 & 2 \\ 4 & 6 & 2 & 8 \\ 8 & 2 & 14 & -4 \\ 2 & 8 & -4 & 14 \end{pmatrix} = P$$

$$\vec{p} = P\vec{b} = \frac{1}{20}\begin{pmatrix} 6 & 4 & 8 & 2 \\ 4 & 6 & 2 & 8 \\ 8 & 2 & 14 & -4 \\ 2 & 8 & -4 & 14 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 16/20 \\ 24/20 \\ 8/20 \\ 32/20 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 6/5 \\ 2/5 \\ 8/5 \end{pmatrix}$$

5. (*10 points.*) **Least Squares**

Putting it all together: Your calculations in Question 4 should help you on this question. **Write** down the equation of the line of best fit (i.e. which has the least square error) through the points **A** $(0,0)$, **B** $(1,1)$, **C** $(2,2)$ and **D** $(-1,1)$ and **calculate** the least square error. Also **Plot** your line of best fit on the graph below.



$$\hat{x} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (A^t A)^{-1} A^t \vec{b} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} \Rightarrow y = \frac{4}{5} + \frac{2}{5}x$$

$$\|\vec{e}\|^2 = \|\vec{p} - \vec{b}\|^2 = \left\| \frac{2}{5}\begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\|^2 = \left\| \begin{matrix} 4/5 \\ 1/5 \\ -3/5 \\ -2/5 \end{matrix} \right\|^2$$

$$= \frac{16}{25} + \frac{1}{25} + \frac{9}{25} + \frac{4}{25}$$

$$= \frac{30}{25} = \frac{6}{5}$$

BONUS (*10 points.*) **Least Squares, Continued.**
Finishing off the problem: Find the equation of the parabola of best fit $y = c_0 + c_1 x + c_2 x^2$ through the points **A** $(0,0)$, **B** $(1,1)$, **C** $(2,2)$ and **D** $(-1,1)$. Give the least square error. Will it be larger or smaller than the least square error you computed in Question 5?

Solve $A^T A \hat{x} = A^T \vec{b}$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ $\quad \hat{x} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 4 \end{pmatrix} \qquad A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 10 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 2 & 6 & 4 \\ 2 & 6 & 8 & 4 \\ 6 & 8 & 18 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & -5 & -5 & -2 \\ 0 & -5 & -3 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1/10 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/10 \\ 0 & 1 & 0 & -1/10 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \qquad y = \frac{3}{10} - \frac{1}{10}x + \frac{1}{2}x^2$$

The least square error should be smaller $\frac{1}{5}$ compared to $4/5$

$$\vec{p} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3/10 \\ -1/10 \\ 5/10 \end{pmatrix} = \begin{pmatrix} 3/10 \\ 7/10 \\ 9/10 \\ 21/10 \end{pmatrix}$$

$$\|\vec{e}\|^2 = \|\vec{p} - \vec{b}\|^2 = \left\| \begin{matrix} 3/10 \\ 3/10 \\ 1/10 \\ 1/10 \end{matrix} \right\|^2 = \frac{9}{100} + \frac{9}{100} + \frac{1}{100} + \frac{1}{100} = \frac{20}{100} = \boxed{\frac{1}{5}}$$