Test 1: Linear Systems

Math 214
Ron Buckmire

Friday March 3 2006
2:30pm-3:25pm

Name: ______________________

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a one hour, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

No.	Score	Maximum
1		20
2		30
3		20
4		30
BONUS		10
Total		**100**

1. Span, Linear Independence, Rank. *20 points.*

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) (*4 points.*) Show that $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) (*4 points.*) Given your knowledge of rref(A), what is the rank of the matrix A?
EXPLAIN YOUR ANSWER.

(c) (*4 points.*) Given your knowledge of rref(A), what is the span of the columns of matrix A? **EXPLAIN YOUR ANSWER.**

(d) (*4 points.*) Given your knowledge of rref(A), discuss the linear independence of the columns of the matrix A. **EXPLAIN YOUR ANSWER.**

(e) (*4 points.*) Given your knowledge of rref(A), discuss whether the matrix A^{-1} exists.
EXPLAIN YOUR ANSWER.

2. Dot product, magnitude, lengths. *30 points.*

Suppose the dot product $\vec{u} \cdot \vec{v}$ is re-defined to be just the product of the lengths of the vectors $\vec{u}$ and $\vec{v}$. Let's call this new dot product the **Buckmire product** and denote it

$$\vec{u} \circ \vec{v} = |\vec{u}||\vec{v}|$$

Discuss which of the following statements are true for all vectors $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$ under the **Buckmire product.**

(a) (*6 points.*) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$

(b) (*6 points.*) $(c\vec{u}) \circ \vec{v} = c(\vec{u} \circ \vec{v})$

(c) (*6 points.*) $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$

(d) (*6 points.*) $\vec{u} \circ \vec{u} \geq 0$

(e) (*6 points.*) $\vec{u} \circ \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$.

3. Matrix Operations. *20 points.*

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

Recall the zero matrix $\mathcal{O}$ and identity matrix $\mathcal{I}$ have particular properties in matrix arithmetic which often (but not always!) correspond to the properties the number zero and the number one that you know and love.

NOTE: A is assumed to be a generic (unknown) $m \times n$ matrix for every part below.

(a) *5 points.* **TRUE or FALSE?** "If $A^2 = \mathcal{O}$ then $A = \mathcal{O}$."

(b) *5 points.* **TRUE or FALSE?** "If $A = \mathcal{O}$ then $A^2 = \mathcal{O}$."

(c) *5 points.* **TRUE or FALSE?** "If $A^2 = \mathcal{I}$ then $A = \mathcal{I}$."

(d) *5 points.* **TRUE or FALSE?** "If $A = \mathcal{I}$ then $A^2 = \mathcal{I}$."

4. Parametric equations, lines, planes, subspaces. *30 points.*

Consider the object described parametrically by $x = t + 1, y = 2t - 3, z = 3t$ in $\mathbb{R}^3$.

(a) (*10 points.*) Write down a system of three linear equations which has this object as its solution.

(b) (*10 points.*) What is the dimension of this object? What is the geometric interpretation of this solution to your linear system in **(a)**? Write down a vector equation describing this object.

(c) (*10 points.*) Is this object a subspace of $\mathbb{R}^3$? **Prove your answer!**

BONUS QUESTION. Linear Independence, Dependence, Inverse. (*10 points.*)
If possible, write down five different 3×3 matrices each one which has one of the following properties:
(i) MATRIX A: The rows are linearly independent but the columns are linearly independent.
(ii) MATRIX B: The rows are linearly dependent but the columns are linearly independent.
(iii) MATRIX C: The rows are linearly independent but the columns are linearly dependent.
(iv) MATRIX D: The rows are linearly dependent but the columns are linearly dependent.
(v) MATRIX E: The transpose of the matrix equals the inverse of the matrix.
EXPLAIN YOUR ANSWER THOROUGHLY. EXTRA CREDIT POINTS ARE HARD TO GET.