# Test 1: Linear Systems

Math 214
Ron Buckmire

Friday March 3 2006
2:30pm-3:25pm

Name: Key

**Directions**: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a one hour, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

| No. | Score | Maximum |
|---|---|---|
| 1 | | 20 |
| 2 | | 30 |
| 3 | | 20 |
| 4 | | 30 |
| BONUS | | 10 |
| **Total** | | **100** |

**1. Span, Linear Independence, Rank.** *20 points.*

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

**(a)** (*4 points.*) Show that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**(b)** (*4 points.*) Given your knowledge of rref($A$), what is the rank of the matrix $A$? **EXPLAIN YOUR ANSWER.**

rank is 3 (number of non-zero rows of matrix A)

**(c)** (*4 points.*) Given your knowledge of rref($A$), what is the span of the columns of matrix $A$? **EXPLAIN YOUR ANSWER.**

span of the columns = $\mathbb{R}^3$ since the columns must be lin ind since the rank = number of columns and no free variables

**(d)** (*4 points.*) Given your knowledge of rref($A$), discuss the linear independence of the columns of the matrix $A$. **EXPLAIN YOUR ANSWER.**

Columns (and rows) are linearly independent

**(e)** (*4 points.*) Given your knowledge of rref($A$), discuss whether the matrix $A^{-1}$ exists. **EXPLAIN YOUR ANSWER.**

$A^{-1}$ exists since rref(A) = I.

**2. Dot product, magnitude, lengths.** *30 points.*

Suppose the dot product $\vec{u}\cdot\vec{v}$ is re-defined to be just the product of the lengths of the vectors $\vec{u}$ and $\vec{v}$. Let's call this new dot product the **Buckmire product** and denote it

$$\vec{u} \circ \vec{v} = |\vec{u}||\vec{v}|$$

Discuss which of the following statements are true for all vectors $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$ under the **Buckmire product.**

(a) (*6 points.*) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$ TRUE

Multiplication is commutative

This statement is true for dot product AND the Buckmire product.

(b) (*6 points.*) $(c\vec{u}) \circ \vec{v} = c(\vec{u} \circ \vec{v})$ FALSE

$(c\vec{u}) \circ \vec{v} = |c\vec{u}||\vec{v}|$

$= |c||\vec{u}||\vec{v}| \neq c|\vec{u}||\vec{v}|$ if $c<0$

This statement is true for dot product, FALSE for Buckmire product

(c) (*6 points.*) $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$ FALSE

$\vec{u} \circ (\vec{v}+\vec{w}) = |\vec{u}||\vec{v}+\vec{w}|$

$|\vec{v}+\vec{w}| \neq |\vec{v}|+|\vec{w}|$ for all $\vec{v}, \vec{w}$

True for dot product, not true for Buckmire product

(d) (*6 points.*) $\vec{u} \circ \vec{u} \geq 0$ TRUE

$\vec{u} \circ \vec{u} = |\vec{u}||\vec{u}| = |\vec{u}|^2 \geq 0$

This is TRUE for Buckmire <u>and</u> dot product.

(e) (*6 points.*) $\vec{u} \circ \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$. TRUE

If $\vec{u}=0 \Rightarrow |\vec{u}|=0 \Rightarrow \vec{u}\circ\vec{u} = |\vec{u}||\vec{u}| = 0\cdot 0 = 0$

If $|\vec{u}|^2 = 0 \Rightarrow |\vec{u}| = 0 \Rightarrow \vec{u} = 0$

TRUE for Buckmire product,

~~not~~ TRUE for dot product also

**3. Matrix Operations.** *20 points.*

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

Recall the zero matrix $\mathcal{O}$ and identity matrix $\mathcal{I}$ have particular properties in matrix arithmetic which often (but not always!) correspond to the properties the number zero and the number one that you know and love.

**NOTE:** $A$ is assumed to be a generic (unknown) $m \times n$ matrix for every part below.

(a) *5 points.* **TRUE or FALSE?** "If $A^2 = \mathcal{O}$ then $A = \mathcal{O}$."

$\boxed{\text{FALSE}}$ If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{O}$

There exists an $A \neq \mathcal{O}$ for which $A^2 = \mathcal{O}$

(b) *5 points.* **TRUE or FALSE?** "If $A = \mathcal{O}$ then $A^2 = \mathcal{O}$."

$\boxed{\text{FALSE}}$ $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A^2$ is not defined

If $A$ is square, then ~~if A = [illegible]~~

$A^2 = A = \mathcal{O}$

(c) *5 points.* **TRUE or FALSE?** "If $A^2 = \mathcal{I}$ then $A = \mathcal{I}$."

$\boxed{\text{FALSE}}$ If $A = A^{-1}$ then $A \cdot A^{-1} = A^2 = I$

A matrix whose inverse equals itself is $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(d) *5 points.* **TRUE or FALSE?** "If $A = \mathcal{I}$ then $A^2 = \mathcal{I}$."

$\boxed{\text{TRUE}}$ $A = I \Rightarrow AA = I \cdot I = I$ ✓

**4. Parametric equations, lines, planes, subspaces.** *30 points.*

Consider the object described parametrically by $x = t + 1, y = 2t - 3, z = 3t$ in $\mathbb{R}^3$.

(a) (*10 points.*) Write down a system of three linear equations which has this object as its solution.

$x = 1 + t \Rightarrow t = x - 1$

$y = 2t - 3 \Rightarrow t = \frac{y+3}{2}$

$z = 3t \qquad t = \frac{z}{3}$

$x - 1 = \frac{y+3}{2}$, $2x - 2 = y + 3$, $\boxed{2x - y = 5}$

$\frac{z}{3} = x - 1$, $z = 3x - 3$, $\boxed{-3x + z = -3}$

$\frac{z}{3} = \frac{y+3}{2}$, $2z = 3y + 9$, $\boxed{2z - 3y = 9}$

(b) (*10 points.*) What is the dimension of this object? What is the geometric interpretation of this solution to your linear system in (a)? Write down a vector equation describing this object.

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & 1 \\ 0 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 9 \end{pmatrix}$$

This is the intersection of three planes in $\mathbb{R}^3$ along a line not thru origin

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t+1 \\ 2t-3 \\ 3t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

This is a 1-D object (a line).

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ -3 & 0 & 1 & -3 \\ 0 & -3 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ -1 & -1 & 1 & 2 \\ 0 & -3 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & +2 & 9 \\ 0 & -3 & 2 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(c) (*10 points.*) Is this object a subspace of $\mathbb{R}^3$? **Prove your answer!**

If $t = 0$, $\vec{x} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \neq \vec{0}$

This object is NOT a subspace of $\mathbb{R}^3$ since the zero vector is NOT on the line.

$$c\vec{x} = \underbrace{ct\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} c \\ -3c \\ 0 \end{pmatrix}}_{\text{Not on the line}}$$

It is also NOT closed under scalar ~~add~~ multiplication or vector addition.

$$\vec{x} + \vec{y} = s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = (s+t)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$$

is in same direction but not on line!

**BONUS QUESTION. Linear Independence, Dependence, Inverse.** (*10 points.*)
If possible, write down five different $3 \times 3$ matrices each one which has one of the following properties:
(i) MATRIX A: The rows are linearly independent but the columns are linearly independent.
(ii) MATRIX B: The rows are linearly dependent but the columns are linearly independent.
(iii) MATRIX C: The rows are linearly independent but the columns are linearly dependent.
(iv) MATRIX D: The rows are linearly dependent but the columns are linearly dependent.
(v) MATRIX E: The transpose of the matrix equals the inverse of the matrix.
**EXPLAIN YOUR ANSWER THOROUGHLY. EXTRA CREDIT POINTS ARE HARD TO GET.**

(i) Use example from Question 1.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

rref(A) = I so columns & rows are lin indep

(ii) $$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

rows are lin ind

(iii) $$B = \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{2} & \cancel{2} & 2 \\ \cancel{3} & \cancel{5} & \cancel{7} \end{pmatrix} \quad C^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

(iv) $$D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

cols & rows are all the same so you have lin dep rows and cols

(v) $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E$$ (or any permutation matrix)