Linear Systems: FINAL EXAM

Wednesday May 10, 2006: 8:30-11:30am — Math 214

Name: Key

Prof. R. Buckmire

Directions: Read *all* problems first before answering any of them. There are EIGHT (8) problems. They are NOT related.

This exam is a limited-notes, closed-book, test. You may use a calculator and bring in one 8.5 inch x 11 inch sheet of paper.

You must include ALL relevant work to support your answers. Use complete English sentences where possible and CLEARLY indicate your final answer from your "scratch work.'

Pledge: I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1.		**25**
2.		**25**
3.		**25**
4.		**25**
5.		**25**
6		**25**
7.		**25**
8.		**25**
TOTAL		**200**

1. *[25 points total.]* **TRUE or FALSE**. TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE.

(a) If $\vec{x}$ and $\vec{y}$ are orthogonal, then they are linearly independent.

$\boxed{\text{TRUE}}$

$\vec{x}\cdot\vec{y} = 0 \Leftrightarrow \vec{x}^T y = 0$ ($\vec{x}$ and $\vec{y}$ are orthogonal)

Proof by contradiction

If $\vec{y} = c\vec{x}$ then $\vec{x}\cdot(c\vec{x}) = c|\vec{x}|^2 = 0$ only if $c = 0$ or $|\vec{x}| = 0 \Leftrightarrow \vec{x} = \vec{0}$.

If $c = 0$ then $\vec{y} = \vec{0}$, and $\vec{x}$ and $\vec{y}$ will be linearly dependent

I think its fair to assume $\vec{x}$ and $\vec{y}$ are non-zero vectors.

If so, then there's no way for a linearly dependent vectors to be orthogonal.

NOT linearly ~~ind~~dependent $\Rightarrow$ not orthogonal

linearly dependent $\Leftarrow$ orthogonal

(b) The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$\boxed{\text{FALSE}}$

row space of $A \subset \mathbb{R}^n$ when A is $m \times n$ matrix

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$ $m = 4$, $n = 2$ $\quad \text{row}(A) \subset \mathbb{R}^2$ not $\mathbb{R}^4$.

(c) $A^T A$ is always a square matrix.

$\boxed{\text{TRUE}}$

A is $m \times n$

A^T is $n \times m$

$A^T A$ is $(n \times m)(m \times n) = n \times n$

(d) The determinant of (A + I) is the determinant of A plus the determinant of I, i.e. $|A + I| = |A| + 1$.

$\boxed{\text{FALSE}}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A + I = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$

$|A| = -2$ $\quad |A + I| = 10 - 6 = 4 \neq -2 + 1$

(e) If P is a symmetric orthogonal matrix, then $P^3 = P$.

$\boxed{\text{TRUE}}$

$\cancel{P^2 = P}$

$P^T P = I$

$P^T = P \Rightarrow P^T P = P^2 = I$

$P^2 \cdot P = I \cdot P$

$P^3 = P$ ✓

2. *[25 points total.]* **Matrix Multiplication.**
In each case write down an example of the required 3×3 matrix being asked for, **and** show the matrix multiplication which produces the desired result for all values of x, y and z. If the required matrix doesn't exist say why.

(a) (*5 points.*) Find the matrix A such that $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}$$

(b) (*5 points.*) Find the matrix B such that $B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z+x \end{bmatrix}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+z \end{pmatrix}$$

(c) (*5 points.*) Find the matrix C such that $C \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 0 \\ 0 \end{pmatrix}$$

(d) (*5 points.*) Find the matrix D such that $D \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(e) (*5 points.*) Find the matrix E such that $E \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$.

No such matrix!

$$E = \begin{pmatrix} -2/x & 0 & 0 \\ 0 & 1/y & 0 \\ 0 & 0 & 3/z \end{pmatrix}$$

but x, y, z might be zero, so $-\frac{2}{x}, \frac{1}{y}, \frac{3}{z}$ don't exist.

3. *[25 points total.]* **Elimination and Inverses.**

In order to solve a linear system $A\vec{x} = \vec{b}$ we end up with the following augmented matrix $\left[\begin{array}{c|c} A & \vec{b}\end{array}\right]$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

(a) (*5 points.*) Find the solution of the linear system, $\vec{x}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$x = 2, y = 1, z = 3$$

(b) (*10 points.*) Find the inverse of the coefficient matrix, A^{-1} by Gauss-Jordan Elimination.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 2 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}$$

(c) (*10 pts.*) Confirm your answers in parts (a) and (b) by computing $A^{-1}A$ and $A^{-1}\vec{b}$.

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} = \begin{matrix} 5 - 3 \\ 5/2 - 6/4 \\ 6/2 \end{matrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \checkmark$$

4. *[25 points total.]* **Eigenvalues, Eigenvectors, Diagonalization.**

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$ where k is an unknown parameter.

(a) (*5 points.*) Find the eigenvalues of the matrix A.

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & K-\lambda \end{vmatrix} = (1-\lambda)(K-\lambda) = 0$$

$\lambda = 1$ and $\lambda = K$

(b) (*10 points.*) Find the eigenvectors of the matrix A.

$$\text{null}(A - I) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & K-1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_2 = 0$
x_1 any

$E_1 = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\text{null}(A - KI) = \left(\begin{array}{cc|c} 1-K & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$(1-K)x_1 + x_2 = 0$
$x_2 = -(1-K)x_1$
$= (K-1)x_1$

$E_K = \text{span}\begin{pmatrix} 1 \\ K-1 \end{pmatrix}$

(c) (*10 points.*) For what values of k can the matrix A be diagonalized, i.e. written as $A = S\Lambda S^{-1}$? If possible, diagonalize the matrix.

If $K=1$ the matrix can NOT be diagonalized, since geometric multiplicity of $\lambda = 1$ would be 1, but algebraic multiplicity is 2.

$K \neq 1$

$$\begin{pmatrix} 1 & 1 \\ 0 & K \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & K-1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} 1 & \frac{-1}{K-1} \\ 0 & \frac{1}{K-1} \end{pmatrix} = S\Lambda S^{-1}$$

Check

$$\underset{A}{\begin{pmatrix} 1 & 1 \\ 0 & K \end{pmatrix}}\underset{S}{\begin{pmatrix} 1 & 1 \\ 0 & K-1 \end{pmatrix}} \stackrel{?}{=} \underset{S}{\begin{pmatrix} 1 & 1 \\ 0 & K-1 \end{pmatrix}}\underset{\Lambda}{\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}} \Rightarrow \begin{pmatrix} 1 & K \\ 0 & K(K-1) \end{pmatrix} \stackrel{\checkmark}{=} \begin{pmatrix} 1 & K \\ 0 & (K-1)K \end{pmatrix}$$

5. *[25 points total.]* **Orthogonalization**

The linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the columns of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$.

(a) (*15 points.*) Use Gram-Schmidt Orthogonalization to convert A into an orthogonal matrix Q whose columns are orthonormal vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3$. [HINT: this will involve computing and not simplifying a number of square roots!]

$\vec{v}_1$ and $\vec{v}_2$ are orthogonal $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2 + 0 - 2 = 0$ ✓

$\vec{v}_1$ and $\vec{v}_3$ are orthogonal $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = -2 + 2 + 0 = 0$ ✓

$\vec{v}_2$ and $\vec{v}_3$ are NOT orthogonal $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = -4 + 0 + 0 \neq 0$

Need to make $\vec{v}_3$ orthogonal to $\vec{v}_1$ and $\vec{v}_2$

$$V_3 = \vec{v}_3 - \operatorname{proj}_{\vec{v}_1}(\vec{v}_3) - \operatorname{proj}_{\vec{v}_2}(\vec{v}_3)$$

$$= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - 0 - \frac{(-4)}{(5)} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \frac{4}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ 1 \\ 4/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$$

(b) (*10 points.*) Compute $Q^T Q$.

$$Q = \begin{pmatrix} \cancel{-2/\sqrt{5}} & 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ \cancel{1/\sqrt{5}} & 2/3 & 0 & 5/\sqrt{45} \\ \cancel{0} & -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -2/\sqrt{45} & 5/\sqrt{45} & 4/\sqrt{45} \end{pmatrix}$$

$$Q^T Q = \begin{pmatrix} 1/9 + 4/9 + 4/9 & 2/3\sqrt{5} - 2/3\sqrt{5} & \frac{-2}{3\sqrt{45}} + \frac{10}{3\sqrt{45}} - \frac{8}{3\sqrt{45}} \\ 2/3\sqrt{5} - 2/3\sqrt{5} & 4/5 + 1/5 & -4/\sqrt{225} + 4/\sqrt{225} \\ -2/3\sqrt{45} + \frac{10-8}{3\sqrt{45}} & -4/\sqrt{225} + 4/\sqrt{225} & \frac{4}{45} + \frac{25}{45} + \frac{16}{45} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. *[25 points total.]* **Linear Independence.**

a. (*10 points*). **If a matrix has more rows than columns, then its columns must be linearly dependent.** Prove this statement is either TRUE or FALSE.

A is $m \times n$ with $m > n$

If we have n vectors in $\mathbb{R}^m$ and $n < m$ then this is less than the number required for a basis for $\mathbb{R}^m$ (i.e. m).

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{These vectors are linearly independent}}$$

FALSE

b. (*10 points*). **If a matrix has more columns than rows, then its columns must be linearly dependent.** Prove this statement is either TRUE or FALSE.

A is $m \times n$ with $n > m$

We have n vectors in $\mathbb{R}^m$ with $n > m$.

Thus since a basis for $\mathbb{R}^m$ contains m vectors and we have more vectors than that in columns of A,

The columns of A MUST be linearly dependent.

TRUE

$$\text{rref}(A) = m\left\{\left(\underbrace{\begin{matrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{matrix}}_{m} \;\; \underbrace{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}}_{n-m}\right)\right.$$

c. (*5 points*). Prove that the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent.

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

only solution is $c_1 = c_2 = c_3 = 0$

Thus the columns of A must be linearly independent.

7. *[25 points total.]* **Projections, Orthogonal Complements.**

Consider $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{bmatrix}$

a. *(6 points)*. Find a basis for the left nullspace of A, i.e. define $N(A^T)$.

$$A^T = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & 2 & -2 \end{pmatrix}$$

$$\text{null}(A^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\text{rref}(A^T) = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

b. *(6 points)*. Find a basis for the column space of A, i.e. define $C(A)$.

$$\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} \right\}$$

c. *(13 points)*. Show that the vector $\vec{x} = \begin{bmatrix} 1 \\ -4 \\ 6 \\ -5 \end{bmatrix}$ can be written as a sum $\vec{x_l} + \vec{x_c}$ where $\vec{x_l}$ is in the left nullspace of A and $\vec{x_c}$ is in the column space of A. Find $\vec{x_l}$ and $\vec{x_c}$.

Recall $(\text{col}(A))^{\perp} = \text{null}(A^T)$

$$\vec{X}_c = \text{proj}_{\text{Col}(A)}(\vec{x}) \qquad \vec{X}_l = \text{proj}_{\text{null}(A^T)}(\vec{x})$$

$$\text{proj}_{\text{col} A}(\vec{x}) = A(A^TA)^{-1}A^T$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{pmatrix} \left[\begin{pmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{pmatrix} \right]^{-1} \begin{bmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

$$= \left(\quad \right) \begin{bmatrix} 30 & 17 \\ 17 & 10 \end{bmatrix}^{-1} \left(\quad \right) = \frac{1}{11} \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 10 & -17 \\ -17 & 30 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & 2 & -2 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} -7 & -3 & -4 & -6 \\ 13 & 4 & 9 & 8 \end{pmatrix}$$

8. *[25 points total.]* **Fundamental Theorem of Linear Algebra.**

Again consider $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{bmatrix}$

a. (*6 points*). Find a basis for the nullspace of A.

$\text{null}(A) = \vec{0}$

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 2 \\ -4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b. (*6 points*). Find a basis for the rowspace of A.

$\text{row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\} = \mathbb{R}^2$

$$P = \frac{1}{11} \begin{pmatrix} 6 & 1 & 5 & 2 \\ 1 & 2 & -1 & 4 \\ 5 & -1 & 6 & -2 \\ 2 & 4 & -2 & 8 \end{pmatrix}$$

$\vec{x}_c = P\vec{x}$

$\vec{x}_\ell = \cancel{P}\vec{x} - \vec{x}_c$

$$\vec{x}_c = \frac{1}{11} \begin{pmatrix} 6 & 1 & 5 & 2 \\ 1 & 2 & -1 & 4 \\ 5 & -1 & 6 & -2 \\ 2 & 4 & -2 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 6 \\ -5 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 22 \\ -33 \\ 55 \\ -66 \end{pmatrix}$$

$$\vec{x}_\ell = \begin{pmatrix} 1 \\ -4 \\ 6 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_c = \begin{pmatrix} 2 \\ -3 \\ 5 \\ -6 \end{pmatrix}$$

$$\vec{x}_\ell \cdot \vec{x}_c = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \\ -6 \end{pmatrix} = -2 + 3 + 5 - 6 = 0 \checkmark$$

c. *(8 points).* Write down the dimensions of each of the four fundamental subspaces of A.

$\dim \operatorname{col}(A) = 2$

$\dim \operatorname{row}(A) = 2$

$\dim \operatorname{null}(A^T) = 2$

$\dim \operatorname{null}(A) = 0$

d. *(5 points).* Confirm the fundamental theorem of algebra showing the appropriate subspaces have the correct relationships between their dimensions and bases.

$2+2 = \dim \operatorname{col}(A) + \dim \operatorname{null}(A^T) = 4 = \#$ of rows

$2+0 = \dim \operatorname{row}(A) + \dim \operatorname{null}(A) = 2 = \#$ of columns

$(\operatorname{col}(A))^\perp = \operatorname{null} A^T$

$\operatorname{row}(A)^\perp = \operatorname{null}(A)$

$\mathbb{R}^2$ is the orthogonal complement of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} = -1-2+3=0$ ✓

$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} = -1-1+2=0$ ✓

$\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} = 4-4=0$ ✓

$\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} = +2-2=0$ ✓

bases are orthogonal