Report on Test 2

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Point Distribution (N=21)

Range	90+	85+	80+	75+	70+	65+	60+	55+	50+	45+	40+	35+	30-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	0	2	1	4	2	4	0	1	1	2	2	1	1

Summary Overall class performance was not stellar. The average score was 62. The high score was 86.

#1 Fundamental Subspaces of a Matrix. Since the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ is symmetric, it's not too surprising to find out that col(A) = row(A) and that null(A) = null(A^T). According to the Fundamental Theorem of Linear Algebra, dim(null(A)) + rank = 3 and since the rank is 2 the nullspaces are one dimensional subspaces of $\mathbb{R}^3$ and row space and column space are two-dimensional subspaces of $\mathbb{R}^3$. The first are spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and the second are spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Notice every vector in one of these spaces is orthogonal to each other, and even the two vectors that span the "larger" space are orthogonal to each other!

#2 Eigenvalues, Eigenvectors, Eigenspaces. To find the eigenspaces of a matrix first one needs to find the eigenvalues, which solve the characteristic polynomial. To find the characteristic polynomial involves computing the determinant, $\det(A - \lambda I) = -\lambda^3 + 2 = 0$. This gives 3 distinct eigenvalues 0, $\sqrt{2}$, and $-\sqrt{2}$. Again, since the matrix is symmetric, we know the eigenvectors will be orthogonal to each other. Also, since one of the eigenvectors is zero, E_0 = null(A) which we know from question 1 is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.
$E_{\sqrt{2}} = \text{span} \left\{ \begin{bmatrix} -1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \right\}$ and $E_{-\sqrt{2}} = \text{span} \left\{ \begin{bmatrix} -1 \\ \sqrt{2} \\ 1 \end{bmatrix} \right\}$. A point to note here is that since these three vectors are orthogonal, they must be linearly independent, so together they form a basis for $\mathbb{R}^3$, since they consist of 3 linearly independent vectors in $\mathbb{R}^3$.

#3 Definition of Subspace. The definition of the vector space $\mathcal{V}$ may appear scary at first, but look at it closely. In English the symbols mean "The set of all vectors in $\mathbb{R}^3$ where the sum of the first and third components of each vector equals zero." **(a)** To prove $\mathcal{V}$ is a subspace one has to check that the set of all vectors which satisfy this definition 1) contains the zero vector; 2) still satisfies the definition when any two of them are added together (i.e. closed under vector addition); and 3) still satisfies the definition when any vector from the set is multiplied by an unknown scalar (i.e. closed under scalar multiplication). **(b)** Notice that the basis vectors in $E_{\sqrt{2}}$ and $E_{-\sqrt{2}}$ as well as in col(A) and row(A) all obey the definition that the sum of the first and third components equal zero! Thus if you want to identify $\mathcal{V}^\perp$ you could use E_0 from Question 2 or null(A) = null(A^T) from Question 1. This space is the set of all vectors whose first and third components are equal while the second component is equal to zero. **(c)** Since bases for $\mathcal{V}$ and identical subspaces have two vectors in their bases, it is a two-dimensional subspace of $\mathbb{R}^3$, i.e. a plane through th origin. It's orthogonal complement, $\mathcal{V}^\perp$, must therefore be a line through the origin, a 1-dimensional subspace of $\mathbb{R}^3$.

#4 Orthogonalization, Normalization. (a) Given a collection of vectors how do we know whether it is a basis for a subspace? A basis for a k-dimensional space has to be comprised of k linearly independent vectors from that space. We know $\mathcal{V}$ is two-dimensional, and there are two vectors in the proposed basis. Both vectors in the basis are elements of the space since they satisfy the definition that their sum of first and third components equal zero. The two vectors are not scalar multiples of one another, so they are linearly independent. Thus the proposed basis is indeed a basis for $\mathcal{V}$. **(b)** An *orthogonal* basis for $\mathcal{V}$ would have to be two vectors which satisfy all the other basis properties but also equal zero when their dot product is taken, for example, $E_{\sqrt{2}}$ and $E_{-\sqrt{2}}$ from Question 2, or the basis for the row space and column space found in rref(A) from Question 1. **(c)** An orthonormal basis is one in which all the vectors are orthogonal to each other and have magnitude one. Since, there's only one vector in a basis for $\mathcal{V}^\perp$ all one needs to do is normalize it (divide by its magnitude).

#5 Orthogonal decomposition, Projection. Now our goal is revealed, to decompose a random vector, $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ into it's components in 3 orthogonal directions, where some of those directions lay in $\mathcal{V}$ and some in $\mathcal{V}^\perp$. We know that if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis then any vector $\vec{b}$ can be written as $\sum_{i=1}^{3} \text{proj}_{\vec{v}_i}(\vec{b})$. We also know that $\vec{b} = \text{proj}_{\mathcal{V}}(\vec{b}) + \text{proj}_{\mathcal{V}^\perp}(\vec{b})$. Since $\mathcal{V}$ is 2-D and $\mathcal{V}^\perp$ is 1-D it's easier to do the projection of $\vec{b}$ onto $\mathcal{V}^\perp$ and THEN find $\text{proj}_{\mathcal{V}}(\vec{b})$ by subtracting. If one uses $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ for a basis for $\mathcal{V}^\perp$ then $\text{proj}_{\mathcal{V}^\perp}(\vec{b}) =$

$$\frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\text{proj}_{\mathcal{V}}(\vec{b}) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. This makes sense, since this last vector does have the property that the sum of its first and third components equals zero, thus it IS in $\mathcal{V}$. However, to write $\vec{b}$ as asum of projections in 3 directions, we need to find projections of $\vec{b}$ in the directions of two vectors which form an **orthogonal** basis for $\mathcal{V}$ and add this to the projection we already have in the $\mathcal{V}^\perp$ direction. Luckily, we were asked for an orthogonal basis for $\mathcal{V}$ in question 4(b). It turns out that

$$\text{proj}_{\mathcal{V}}(\vec{b}) = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This means that $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Clearly $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$ with the first vector forming a basis for $\mathcal{V}^\perp$ and the last two forming an orthogonal basis for $\mathcal{V}$.