
Multivariable Calculus

Math 212 Spring 2015

Fowler 309 MWF 9:35am - 10:30am

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<http://faculty.oxy.edu/ron/math/212/15/>

Worksheet 24

TITLE Vector Fields and the Line Integral

CURRENT READING McCallum, Section 17.3, 18.1-18.2)

HW #10 (DUE Thursday 04/16/15 by 5PM)

McCallum, *Section 17.2*: 1, 4, 10, 14, 27, 28, 36.

McCallum, *Chapter 17.3*: 1, 12, 18, 21, 28, 29.

McCallum, *Chapter 17 Review*: 2, 4, 5, 7, 10, 12, 18, 20, 32, 33, 34.

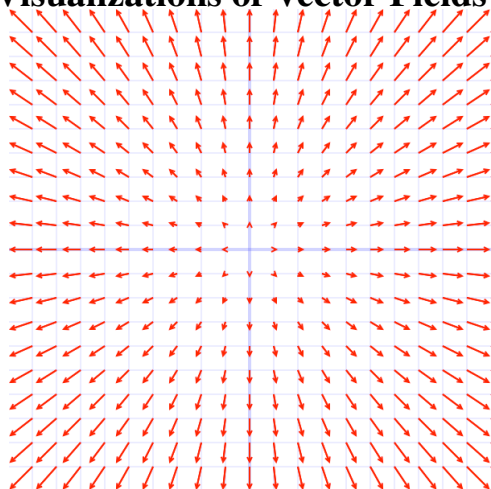
SUMMARY

This worksheet introduces the idea of vector fields, the concept of a line or path integral and the algorithm for how to evaluate line integrals given a path.

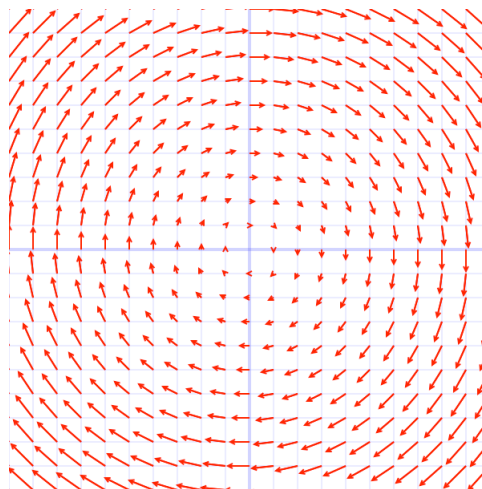
DEFINITION: vector field in \mathbb{R}^2

A **vector field** in \mathbb{R}^2 is a function $\vec{F}(x, y)$ which produces as its output a 2-dimensional vector at every point in the plane. For example, the gradient of a function $\text{grad } f(x, y)$ is a vector which points in the direction of greatest increase of f at every position in the xy -plane.

Visualizations of Vector Fields



$$\vec{F}(x, y) = x\hat{i} + y\hat{j}$$



$$\vec{G}(x, y) = y\hat{i} - x\hat{j}$$

Exercise

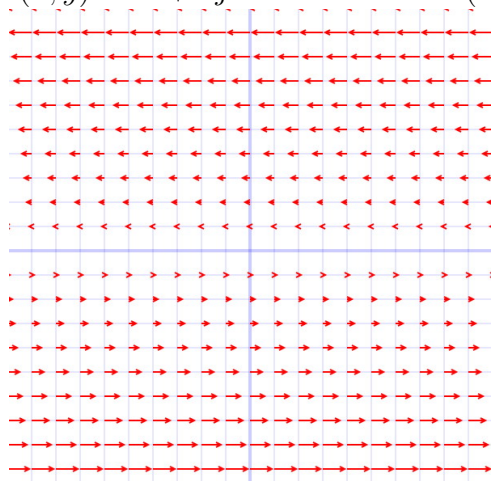
Give a sketch of the vector field $\vec{H}(x, y) = -x\hat{i}$

Exercise

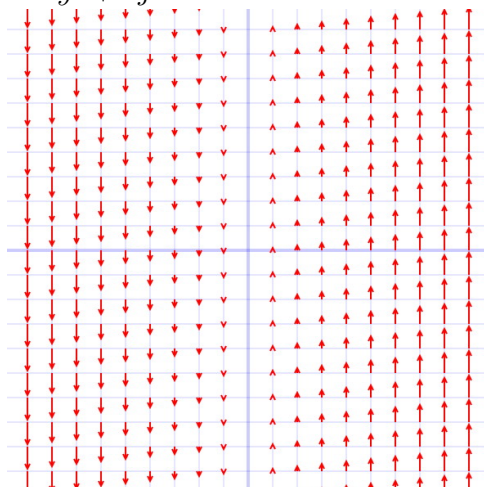
Match the vector field with the corresponding figure.

$$\vec{A}(x, y) = (x + y)\hat{i} + (x - y)\hat{j} \quad \vec{B}(x, y) = \frac{-x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{-y}{\sqrt{x^2 + y^2}}\hat{j}$$

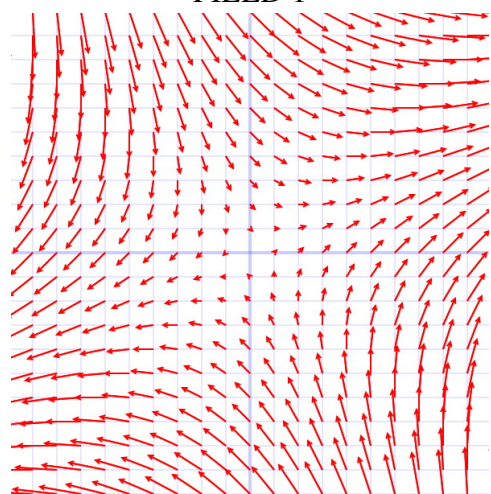
$$\vec{C}(x, y) = 0\hat{i} + x\hat{j} \quad \vec{D}(x, y) = -y\hat{i} + 0\hat{j}$$



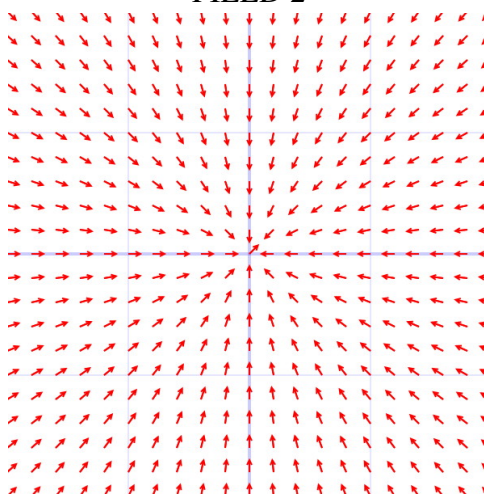
FIELD 1



FIELD 2



FIELD 3



FIELD 4

Gradient Field

If a vector field is formed from taking the gradient of some scalar function f , then such a field is known as a **gradient field** and is given by $\vec{F} = \vec{\nabla}f$.

GROUPWORK

Determine whether any of the given fields $\vec{F} = x\hat{i} + y\hat{j}$, $\vec{G} = -y\hat{i} + x\hat{j}$ or $\vec{H} = x\hat{i}$ are gradient fields.

Path Integrals or Line Integrals

DEFINITION: path integral or line integral

Given a vector field $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a path C in \mathbb{R}^2 given by the parametrization $\vec{g}(t) : \mathbb{R} \rightarrow \mathbb{R}^2$ for $a \leq t \leq b$, the **path integral** of \vec{F} over C is written as $\int_C \vec{F} \cdot d\vec{x}$ and evaluated by computing

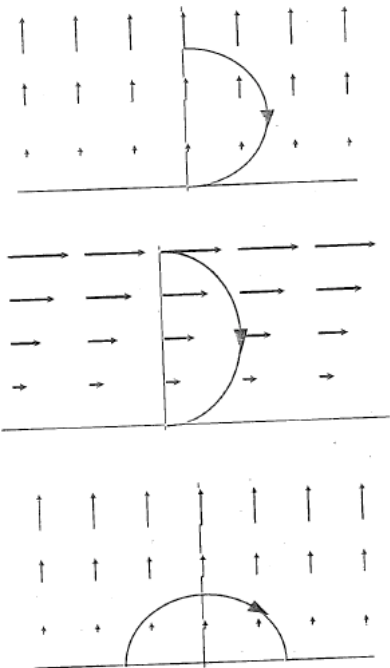
$$\int_a^b \vec{F}(\vec{g}(t)) \cdot \frac{d\vec{g}}{dt} dt.$$

CONCEPTUAL UNDERSTANDING

The meaning of the expression $\int_C \vec{F} \cdot d\vec{x}$ is the amount of work done by a Force $\vec{F}(\vec{x})$ acting along the path C .

EXAMPLE

Consider the following paths and vector fields. Let's use our understanding of path integrals to at least estimate the sign of the line integral $\int_C \vec{F} \cdot d\vec{x}$.



Properties of Path Integrals or Line Integrals

Given a scalar constant λ , vector fields \vec{F} and \vec{G} and oriented curves (i.e. paths) C_1 , C_2 and C the following properties apply to line integrals.

Linearity

$$\int_C (\vec{F} + \vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$$

$$\int_C \lambda \vec{F} \cdot d\vec{r} = \lambda \int_C \vec{F} \cdot d\vec{r}$$

Reversing Orientation

$$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$$

Additivity

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} \vec{F} \cdot d\vec{r}$$

Evaluation of Path Integrals or Line Integrals**ALGORITHM**

This is an algorithm (step-by-step method) for evaluating the line integral $\int_C \vec{F} \cdot d\vec{r}$

STEP 1 Parametrize the path C with a function of your choice $\vec{r}(t)$ for $a \leq t \leq b$

STEP 2 Compute $\frac{d\vec{r}}{dt}$ and $\vec{F}(\vec{r}(t))$ and take their dot product. i.e. $\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$

STEP 3 Integrate the function of t given in Step 2 with respect to t from $t = a$ to $t = b$, i.e. the number $\int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$ is the value of the given line integral

EXAMPLE

McCallum, Example 2, page 959. Given the vector field $\vec{F} = x\hat{i} + y\hat{j}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ when

(a) C_1 is the line segment joining $(1, 0)$ to $(0, 2)$

(b) C_2 is a path formed by part of a parabola with its vertex at $(0, 2)$ joining $(1, 0)$ to $(0, 2)$

(c) C_3 is the closed path formed by following the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, 2)$ traversed in a counter-clockwise fashion.