Multivariable Calculus

Math 212 Spring 2015 — Fowler 309 MWF 9:35am - 10:30am

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Worksheet 22

TITLE The Calculus Of Curves In Space
CURRENT READING McCallum, Section 17.1-17.2
HW #9 (DUE TUESDAY 4/7/15 5PM)
McCallum, *Section 16.3*: 2, 5, 6, 28, 39, 40, 41, 42, 54*,55*.
McCallum, *Chapter 16.4*: 3, 7, 8,17, 20, 22.
McCallum, *Chapter 16.5*: 12, 13, 14, 15, 21, 22, 23, 63*, 73*.
McCallum, *Chapter 16 Review*: 1, 4, 10, 11, 12, 14, 20, 23, 55*, 56*.

SUMMARY
This worksheet discusses curves in space, primarily parametric equations in $\mathbb{R}^3$ of curves in space. In addition, we will learn about the physical significance of derivatives of these vector functions of scalar variables, i.e. $\vec{x}(t)$.

`RECALL:` The equation of a line in space can be written as $\vec{x} = \vec{p} + \vec{d}t$ where $\vec{p}$ is the position vector for a point on the line and $\vec{d}$ is the direction (i.e. displacement) in which points on the line move.

Parametric Equation Of A Line in $\mathbb{R}^3$

Suppose we have linear equations $x = a + bt$, $y = c + dt$, $z = e + ft$ where a, b, c, d, e and f are known numbers

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ c \\ e \end{bmatrix} + t \begin{bmatrix} b \\ d \\ f \end{bmatrix} = \vec{p} + \vec{d}t$$

EXAMPLE

Which of the following set of parametric equations is a representation for a line in $\mathbb{R}^3$?

$$x = 1 - 2t, \quad y = t \quad z = 2 + t^2$$

OR

$$x = 1 - 2t^2, \quad y = t^2 - 4 \quad z = 2 + t^2$$

Exercise

Sketch the curve being described by $\vec{x} = \begin{bmatrix} 2t - 4 \\ t^2 + 3 \end{bmatrix}$ for all $t \in \mathbb{R}$

GROUPWORK

In each caseketch the curve being described by the following parametrizations

(a) $x(t) = t^3, \quad y(t) = t^6$ for $-\infty < t < \infty$

(b) $x(t) = t^2, \quad y(t) = t^4 - 1$ for $0 \le t \le 2$

(c) $x(t) = \cos(t^2), \quad y(t) = 2\sin(t^2)$ for $0 \le t \le \sqrt{\pi}$

(d) $x(t) = t^2, \quad y(t) = t$ for $-2 \le t \le 2$

Parametrized Curves

DEFINITION: parametrization of a curve in $\mathbb{R}^2$

A path traced by a point $P = (x, y)$ where x and y are functions of a parameter t is called a **parametric curve** or parametrized path. We say that $\vec{x} = (x(t), y(t))$ where $\vec{x}$ is the position vector of the points on the curve.

NOTE:

We could expand these parametrization to $\mathbb{R}^n$ if we had n equations in the parametric variable.

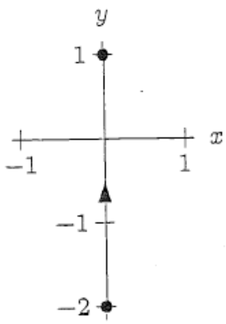
Parametrizations are not unique. Any path can be parametrized in an infinite number of different ways. For example, the curve $y = f(x)$ can always be parametrized by $x = t, y = f(t)$.

The path is different from the curve. The path is traversed in the direction of increasing values of the parameter, while the curve is simply the set of points the path follows.

In $\mathbb{R}^2$ the slope of the tangent line to the parametrized curve at any point $\frac{dy}{dx}$ can be found by the ratio of $\frac{dy}{dt}$ over $\frac{dx}{dt}$ provided that $x(t)$ and $y(t)$ are differentiable functions and $x'(t) \neq 0$.

GroupWork

Find parametrizations for the given curves.

1.


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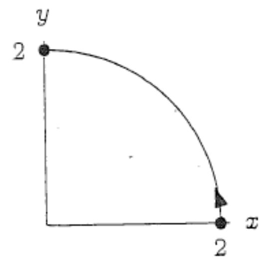

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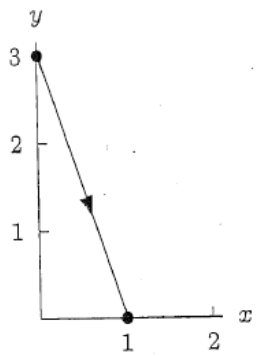

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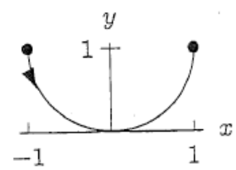

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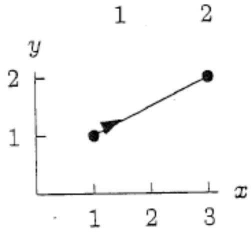

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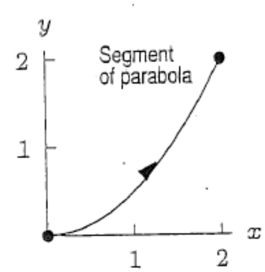

Velocity and Acceleration of Particles

DEFINITION: velocity vector and acceleration vector

Given a position vector $\vec{x}$ as a function of a parameter t for a particle, the velocity vector $\vec{v}$ is given by $\dfrac{d\vec{x}}{dt}$ with the speed being $||\vec{v}||$ and the acceleration vector $\vec{a}$ is given by $\dfrac{d\vec{v}}{dt}$

In other words, if $\vec{x}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, then

$$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

and

$$\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$$

Sometimes $\vec{v}$ and $\vec{a}$ are written $\dot{x}$ and $\ddot{x}$, respectively.

Distance Travelled

The length of a curve given by a position vector $\vec{x}(t)$ defined for an interval $a \leq t \leq b$ if the derivative $\dfrac{d\vec{x}}{dt} \neq 0$ in the interval $a < t < b$ is given by

$$\int_a^b \left|\left|\frac{d\vec{x}}{dt}\right|\right| \, dt \qquad \text{or} \quad \int_a^b ||\vec{v}|| \, dt$$

Circular Motion

Given that $\vec{x}(t) = R\cos(\omega t)\hat{\imath} + R\sin(\omega t)\hat{\jmath}$, this represents motion in a circle of radius R with period $2\pi/|\omega|$ where $\vec{v}$ is tangent to the circle and speed is a constant $|\omega|R$ and acceleration is $\omega^2 R$ pointed towards the center of the circle.

EXAMPLE

Let's show the features of circular motion given above are true.