Multivariable Calculus

Math 212 Spring 2015 | Fowler 309 MWF 9:35am - 10:30am

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Worksheet 21

TITLE Evaluating Multiple Integrals Using Other Coordinate Systems
CURRENT READING McCallum, Section 16.4-16.5, 21.2
HW #9 (DUE TUESDAY 4/7/15 5PM)
McCallum, *Section 16.3*: 2, 5, 6, 28, 39, 40, 41, 42, 54*,55*.
McCallum, *Chapter 16.4*: 3, 7, 8,17, 20, 22.
McCallum, *Chapter 16.5*: 12, 13, 14, 15, 21, 22, 23, 63*, 73.
McCallum, *Chapter 16 Review*: 1, 4, 10, 11, 12, 14, 20, 23, 55*, 56*.

SUMMARY
This worksheet discusses how to compute iterated integrals in other coordinate systems, namely polar coordinates, spherical coordinates and cylindrical coordinates.

RECALL Points in the xy-plane can also be represented by a different coordinate system, called **polar coordinates** where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. In other words,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

The Double Integral In Polar Coordinates

Consider the following integral

$$\int_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta \tag{1}$$

NOTE

Note that the dA which in regular Cartesian coordinates is either $dx\ dy$ or $dy\ dx$ becomes $r\ dr\ d\theta$ NOT simply $dr\ d\theta$!

Some problems are easier to do in polar coordinates than cartesian coordinates!

EXAMPLE

Evaluate $\int_D \cos(x^2 + y^2)\, dA$ where D is the disk (i.e. interior and boundary) of radius $\sqrt{\pi/2}$ centered at $(0,0)$.

THEOREM

Jacobi's Theorem for Transforming Integrals Between Coordinate Systems

The integral of a continuous function $f(\vec{x})$ over a region $\mathcal{W}$ in $\mathbb{R}^n$ can be transformed into an equivalent integral of $f(\vec{T}(\vec{x}))$ in a region $\mathcal{W}^*$ where $\vec{T}$ is a continuously differentiable transformation that maps $\mathcal{W}$ to $\mathcal{W}^*$, i.e. $\mathcal{W}^* = T(\mathcal{W})$.

In other words, suppose in $\mathbb{R}^3$ that $\vec{T}(\vec{x}) = \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z \end{bmatrix}$ so that

$$\int\int\int_W f(x,y,z)\,dx\,dy\,dz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \text{ in } \mathbb{R}^3$$

and in $\mathbb{R}^2$ $\vec{T}(\vec{x}) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ so that

$$\int\int_W f(x,y)\,dx\,dy = \int\int_{W^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \text{ in } \mathbb{R}^2$$

The expressions $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ and $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ are called the **Jacobian** of the transformation. In actuality they are the determinant of the Jacobian matrix associated with the transformation.

DEFINITION: **The Jacobian Matrix**

The Jacobian matrix of a function $\vec{f}(\vec{x})$ is a matrix where the term in the i^{th} row and j^{th} column is the expression $\frac{\partial f_i}{\partial x_j}$ where f_i is the i^{th} component of the vector function$\vec{f}(\vec{x})$ and x_j is the j^{th} component of the vector variable $\vec{x}$.

The Jacobian matrix for $\vec{T}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively is given below

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

CONCEPTUAL UNDERSTANDING

Generally, we use Jacobi's theorem to convert from Cartesian coordinates to polar, spherical, and cylindrical co-ordinates.

Change of Variables: Polar Coordinates

$$\int\int_D f(x,y)\,dx\,dy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

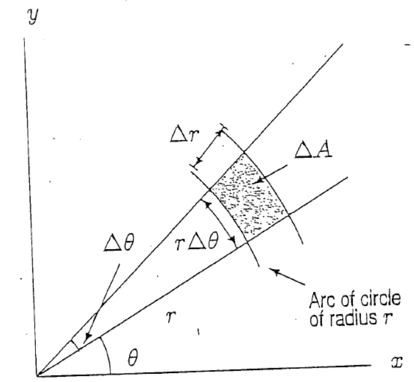
Change of Variables: Cylindrical Coordinates

$$\int\int\int_W f(x,y,z)\,dx\,dy\,dz = \int\int\int_{W^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

Change of Variables: Spherical Coordinates

$$\int\int\int_W f(x,y,z)\,dx\,dy\,dz = \int\int\int_{W^*} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi$$

Visualizing The Area Differential In Polar Coordinates



The area of the segment of the circular arc of radius r of angular width $\Delta\theta$ and length Δr is $\Delta A \approx (r\Delta\theta)(\Delta r)$

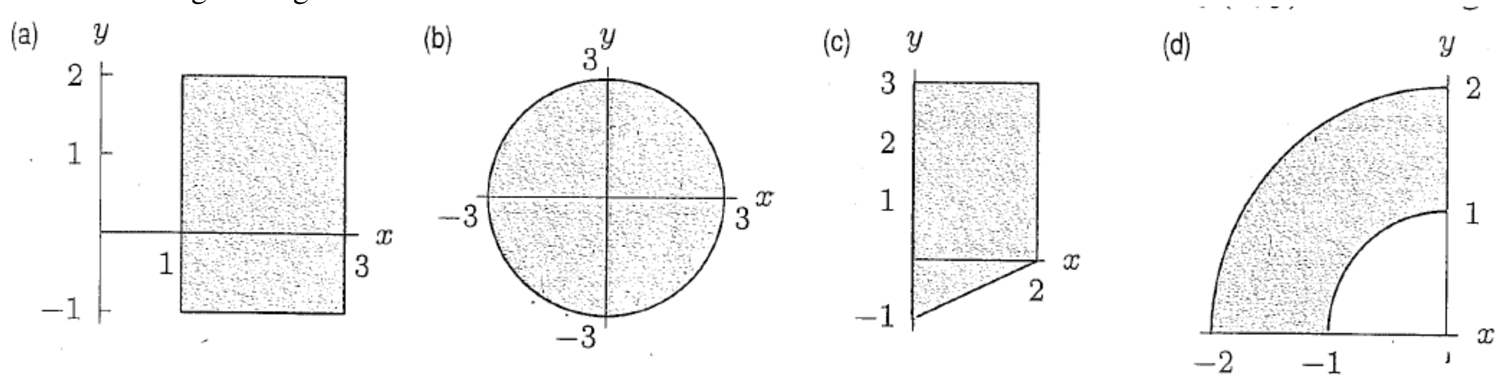
EXAMPLE

We can use the Jacobian of the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) to explain why $dx\, dy = r\, dr\, d\theta$.

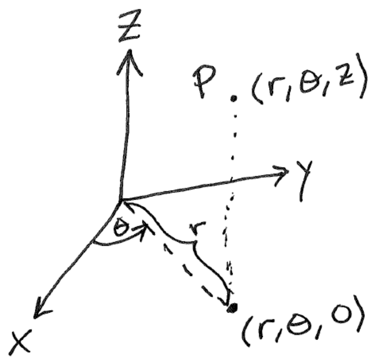
Let $\vec{T}(\vec{x}) = \begin{bmatrix} x(r,\theta) \\ y(r,\theta) \end{bmatrix}$ where $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$ and compute $\left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right|$

Exercise

McCallum, page 893, Example 3. For each of the regions below decide whether to integrate using polar or Cartesian coordinates. Write down an iterated integral of an arbitrary function $f(x, y)$ over the given region.

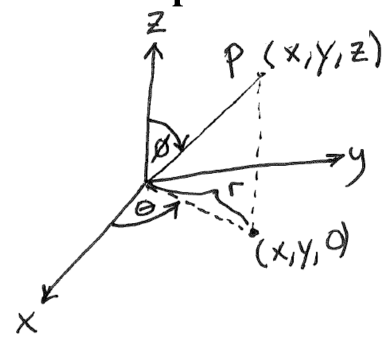


Cylindrical Coordinates



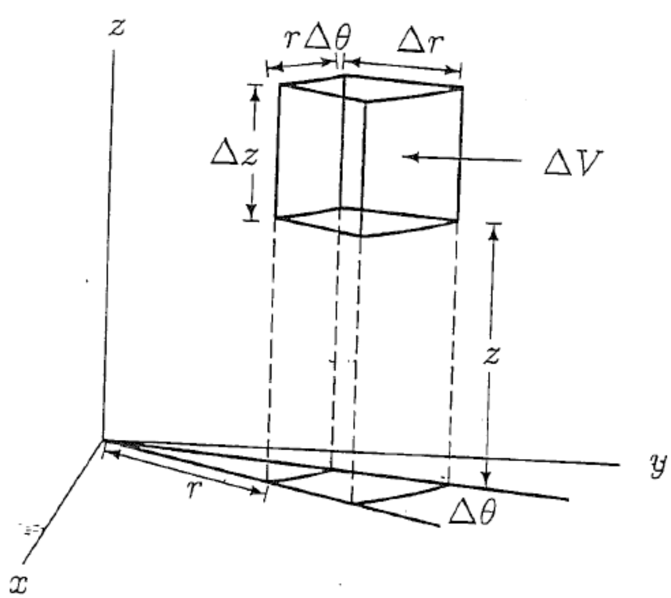
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$0 \le r < \infty,\ -2\pi \le \theta \le 2\pi,\ -\infty < z < \infty$
$r^2 = x^2 + y^2$

Spherical Coordinates

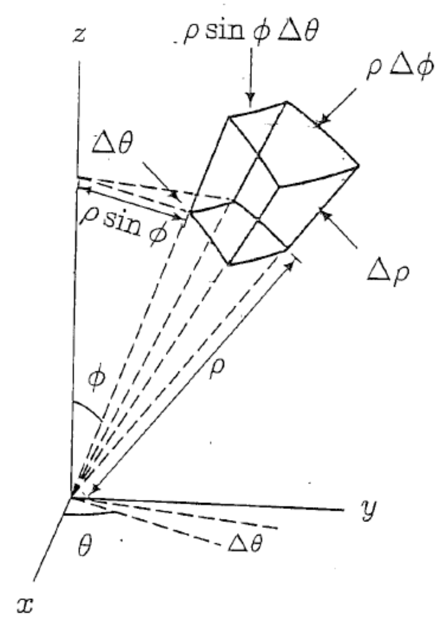


$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\theta$
$0 \le \rho < \infty,\ -2\pi \le \theta \le 2\pi,\ 0 \le \phi \le \pi$
$\rho^2 = x^2 + y^2 + z^2$

Visualizing The Volume Differential In Cylindrical Coordinates and Spherical Coordinates



$\Delta v \approx r\Delta r\Delta z\Delta\theta$



$\Delta V \approx \rho^2\sin\phi\Delta\rho\Delta\theta\Delta\phi$

EXAMPLE

McCallum, page 903, Exercise 19. Write a triple integral in cylindrical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $dz\ dr\ d\theta$.

Evaluate the triple integral to show the volume of the sphere of radius K is $\frac{4}{3}\pi K^3$.

Exercise

McCallum, page 903, Exercise 20. Write a triple integral in spherical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $d\theta\ d\rho\ d\phi$.

Evaluate the triple integral to show the volume of the sphere of radius K is $\frac{4}{3}\pi K^3$.

GROUPWORK

McCallum, page 903, Exercise 24-25.

Use (a) Cartesian (b) Cylindrical (c) Spherical coordinates to write down the limits of integration for $\int_W dV$ for the following figures.

24. One-eighth of the sphere with unit radius centered at the origin (occupying the positive x, y and z quadrants)

25. The shape formed by a cone with 90° vertex at the origin topped by the sphere of radius 1 centered at the origin. (Sort of looks like an ice-cream cone.)