Multivariable Calculus

Math 212 Spring 2015 | Fowler 309 MWF 9:35am - 10:30am

 | http://faculty.oxy.edu/ron/math/212/15/

Worksheet 18

TITLE Integration of a Multivariable Function $f(x, y)$
CURRENT READING McCallum, Section 16.1-16.2
HW #8 (DUE Wednesday 04/01/15 5PM)
McCallum, *Section 16.1*: 2, 4, 6, 7, 8, 14, 22, 23..
McCallum, *Chapter 16.2*: 1, 3, 4, 7, 11, 13, 14, 16, 18, 19, 23, 33, 34, 37, 38, 43, 50.

SUMMARY
This worksheet discusses the concept of the integral of a surface $f(x, y)$ over a region R, known as a double integral which may be used to compute areas and/or volumes.

The Definite Integral $\int_a^b f(x)\,dx$

RECALL

Given a function $f(x)$ defined on an interval $a \leq x \leq b$ the **definite integral** $\displaystyle\int_a^b f(x)dx$ can be defined as

$$\int_a^b f(x)\,dx = \lim_{max \Delta x_k \to 0} \sum_{k=1}^{N} f(x_k)\Delta x_k$$

CONCEPT

The definite integral can be approximated by the limit of the Riemann sums and represents the signed area "under" a curve $y = f(x)$. You can think of this conceptually as a number of rectangles representing the area which in the limit as the width of the rectangles goes to zero (and the number of rectangles goes to infinity) becomes a finite number which is exactly the value of the area.

EXAMPLE

Draw a picture representing the right-hand Riemann sum approximations of the integral of the function $f(x) = x^2 + 1$ over the interval $0 \leq x \leq 3$.

Exercise

What is the average value of the function $f(x) = x^2 + 1$ over the interval $0 \leq x \leq 3$?

The Double Integral $\int\int_{\mathcal{R}} f(x,y)\,dA$

The **double integral** of $f(x,y)$ over the rectangular region $\mathcal{R}$ is defined as

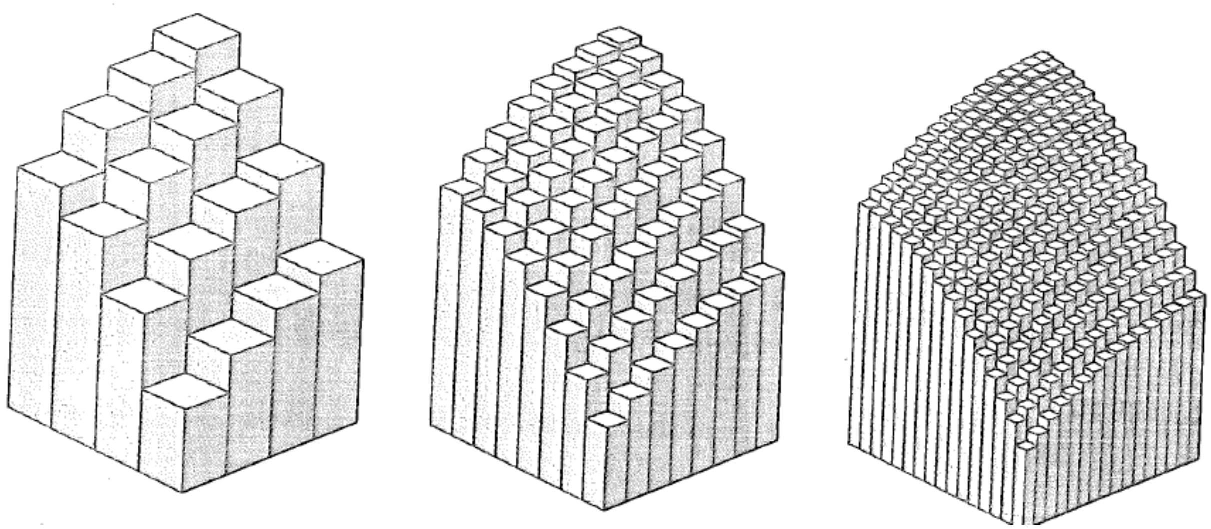
$$\int\int_{\mathcal{R}} f(x,y)\,dA = \lim_{max\Delta A_{ij}\to 0}\sum_{i=1}^{M}\sum_{j=1}^{N} f(x_{ij},y_{ij})\Delta A_{ij}$$

OR

$$\int\int_{\mathcal{R}} f(x,y)\,dA = \lim_{M,N\to\infty}\sum_{i=1}^{M}\sum_{j=1}^{N} f(x_i,y_j)\Delta x_i\Delta y_j$$

CONCEPT

The double integral can be thought of as the three-dimensional analogue to the limit of the Riemann Sums in the single-variable definite integral. This time the double integral is approximated by boxes which have bases of area $\Delta x_i * \Delta y_j$ and height equal to $f(x_i,y_j)$. As the number of boxes becomes infinitely large and the area of the base of each box A_{ij} goes to zero then if the limit exists then we say the function $f(x,y)$ is integrable and the integral represents the volume under the surface $f(x,y)$ above the region $\mathcal{R}$.
A visualization of this concept is depicted in the figure below.



The Double Integral Can Represent Volume

If the function $z = f(x,y)$ is always positive over the specific region $\mathcal{R}$ then the double integral can represent volume.

$$\int\int_{\mathcal{R}} f(x,y)\,dA = \text{Volume under } f \text{ above the region } \mathcal{R}$$

The Double Integral Can Represent Area

If the function $f(x,y) = 1$ then the double integral can represent area.

$$\int\int_{\mathcal{R}} f(x,y)\,dA = \int\int_{\mathcal{R}} 1\,dA = \text{Area of Region } \mathcal{R}$$

Using an idea from single-variable Calculus, the average value $\bar{f}$ of a function over a particular region $\mathcal{R}$ can be found:

$$\bar{f} = \frac{1}{\text{Area of Region } \mathcal{R}}\int\int_{\mathcal{R}} f(x,y)\,dA = \text{Average value of } f \text{ Over Region } \mathcal{R}$$