Multivariable Calculus

Math 212 Spring 2015 Fowler 309 MWF 9:35am - 10:30am
 http://faculty.oxy.edu/ron/math/212/15/

Worksheet 17

TITLE Constrained Multivariable Optimization (Using Lagrange Multipliers)
CURRENT READING McCallum, Section 15.3
HW #7 (DUE THURSDAY 03/26/15 AT 5PM)
McCallum, *Section 15.1*: 4, 13, 20, 21, 25, 32, 37, 40*.
McCallum, *Section 15.2*: 8, 9, 10, 11, 12, 17, 19, 20, 27, 31*, 36.
McCallum, *Section 15.3*: 2, 5, 8, 14, 18, 21, 31, 34, 44*.
McCallum, *Chapter 15 Review*: 12, 23, 24, 25, 26, 41, 44*.

SUMMARY
This worksheet discusses the concept of optimizing a multivariable objective function $f(x, y)$ subject to a multivariable constraint function $g(x, y) = c$. This is a classic problem that is often solved by a technique called using Lagrange multipliers.

Constrained Multivariable Optimization

Oftentimes we want to optimize (find the maximum/minimum of a particular function, called the **objective function** subject to a specific set of conditions, which is called the **constraint**.

EXAMPLE

McCallum, page 850, Example 1.
Find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 1$.

Method of Lagrange Multipliers

A smooth objective function $f(\vec{x})$ has a maximum or minimum subject to a smooth constraint $g(\vec{x}) = c$ at a point $\vec{x}_0$ then either

The point $\vec{x}_0$ satisfies the equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g = c$

OR $\vec{x}_0$ is an endpoint of the constraint g

OR $\vec{\nabla} f(\vec{x}_0) = \vec{0}$

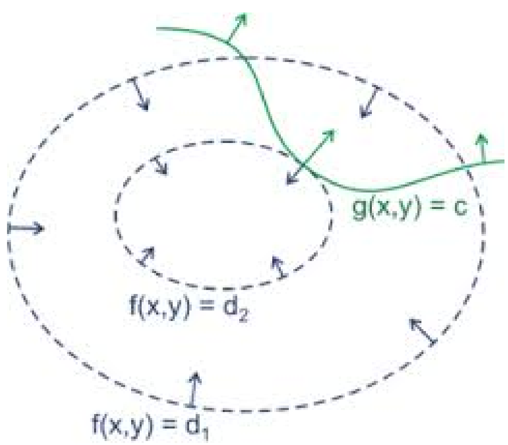
To find $\vec{x}_0$ compare values of the objective function f at the points satisfying each of the above conditions. The number λ is called the **Lagrange multiplier**.

EXAMPLE

McCallum, page 850, Example 1.

Let's find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 1$ using Lagrange Multipliers.

Understanding Why Lagrange Multipliers Work



The reason why the method of Lagrange Multipliers works is related to the meaning of the gradient. We know that the gradient points in the direction of greatest increase of a function $f(x, y)$ and is always orthogonal to level sets of f. In the figure, the little arrows show the direction of $\vec{\nabla} f$ and $\vec{\nabla} g$ and you can see they are perpendicular to the level sets of f and g.

The maximum and minimum value of $f(x, y)$ subject to the constraint $g(x, y) = c$ means that you are looking for a location where a level set of $f(x, y)$ is exactly tangential to the specific level set of $g(x, y) = c$. At this point grad f will be parallel to grad g.

QUESTION: Where in the figure is the location of the extrema of f subject to the constraint $g(x, y) = c$? Is this extremum a minimum or maximum? HOW CAN YOU TELL?

 Algorithm for Multivariable Optimization

Given a function $f(\vec{x})$ defined inside and on a region R in $\mathbb{R}^n$, compare the values of f at the following points:

(a) Critical points of f in the interior of R, where $\vec{\nabla} f = \vec{0}$

(b) Points on the Boundary of R

1. EITHER: Find a parametric representation $\vec{g}$ for the boundary of R, in which case we have a new optimization problem with the composite function $\vec{f}(\vec{g})$ defined on a set of one lower dimension,
2. OR: Use the Lagrange multiplier method by solving the system $\vec{\nabla} f = \sum_{k=1}^{n} \lambda_k \vec{\nabla} g_k$ where g_k are the functions representing the n constraints.

$\boxed{\textbf{Exercise}}$

McCallum, page 852, Example 2.

Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y-2)^2$ subject to the constraint $x^2 + y^2 \leq 45$

Interpreting The Meaning of λ

The value of the Lagrange Multiplier has an actual physical meaning, it is the rate of change of the optimum value of the objective function $f(x, y)$ with respect to the increase in the value of the constraint c, where the constraint function was $g(x, y) = c$.
We can show this by the Chain Rule if we consider the optimum point of $f(x, y)$ under the constraint $g(x, y) = c$ to be the point (x_0, y_0) where x_0 and y_0 are functions of c so that $f(x_0(c), y_0(c))$ is the optimal value and $g(x_0(c), y_0(c)) = c$ is the constraint.

$$\begin{aligned}
\frac{df}{dc} &= \frac{\partial f}{\partial x}\frac{dx_0}{dc} + \frac{\partial f}{\partial y}\frac{dy_0}{dc} \\
&= \left(\lambda\frac{\partial g}{\partial x}\right)\frac{dx_0}{dc} + \left(\lambda\frac{\partial g}{\partial y}\right)\frac{dy_0}{dc} \\
&= \lambda\left(\frac{\partial g}{\partial x}\frac{dx_0}{dc} + \lambda\frac{\partial g}{\partial y}\frac{dy_0}{dc}\right) \\
&= \lambda\frac{dg}{dc} \\
&= \lambda
\end{aligned}$$

GROUPWORK

Adapted from **McCallum, page 856, Exercise 20.** Consider the contours of f in the figure.
(a) Does f have a maximum value subject to the linear constraint function $g(x, y) = c$ for $x \geq 0, y \geq 0$? If so, approximately where is it and what is its value?
(b) Does f have a minimum value subject to the constraint? If so, approximately where is it and what is its value?
(c) Considering that $g(x, y) = c$ is linear in x and y what is the sign of λ? (In what direction does g increase as c increases?)