Multivariable Calculus

Math 212 Spring 2015 | Fowler 309 MWF 9:35am - 10:30am

 http://faculty.oxy.edu/ron/math/212/15/

Worksheet 15

TITLE Local Extrema of a Multivariable Function $f(x, y)$
CURRENT READING McCallum, Section 15.1
HW #7 (DUE THURSDAY 03/26/15 AT 5PM)
McCallum, *Section 15.1*: 4, 13, 20, 21, 25, 32, 37, 40*.
McCallum, *Section 15.2*: 8, 9, 10, 11, 12, 17, 19, 20, 27, 31*, 36.
McCallum, *Section 15.3*: 2, 5, 8, 14, 18, 21, 31, 34, 44*.
McCallum, *Chapter 15 Review*: 12, 23, 24, 25, 26, 41, 44*.

SUMMARY
This worksheet discusses the concept of local extrema (maxima or minima) of functions of multivariable functions. We present the definition of critical points for $f(x, y)$ and introduce the concept of a critical point which is neither a local maximum or local minimum: the saddle point.

DEFINITION: **local maximum** and **local minimum** of a multivariable function

f has a **local maximum** at the point P_0 if $f(P_O) \geq f(P)$, for all points P near P_0.
Similarly, f has a **local minimum** at the point P_0 if $f(P_0) \leq f(P)$, for all points P near P_0.

Local Extrema For a Surface $z = f(x, y)$

The surface $z = f(x, y)$ has a **local maximum** at the point (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$, for all (x, y) in some neighborhood of (x_0, y_0).
The surface $z = f(x, y)$ has a **local minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$, for all (x, y) in some neighborhood of (x_0, y_0).

RECALL: Critical Points of $y = f(x)$

To find the location of the local maximum or local minimum of a single variable function $y = f(x)$ you found candidate points called **critical points** by determining where $f'(c) = 0$ or $f'(c)$ failed to exist. We called the point $(c, f(c))$ a critical point of $f(x)$.

DEFINITION: **critical points** of a multivariable function

The critical points of a multivariable function $f(\vec{x})$ are the points $\vec{c}$ where the gradient function $\vec{\nabla} f$ is either $\vec{0}$ or undefined.

EXAMPLE

McCallum, page 832, Example 2.
Find and analyze any critical points of $f(x, y) = -\sqrt{x^2 + y^2}$

`QUESTION:` What is the difference between a local extrema and a global extrema?

Exercise

McCallum, page 832, Example 3.

Find and analyze any critical points of $g(x, y) = x^2 - y^2$. Does this function have a local maximum or local minimum?

QUESTION: Does every critical point have to be a local maximum or local minimum?

Second Derivative Test For Functions Of Two Variables

Given (a, b) where $f_x(a, b) = 0$ and $f_y(a, b) = 0$ Let

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = \det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

One can classify the point (a, b) according to the value of D

- If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum of $f(x, y)$
- If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum of $f(x, y)$
- If $D < 0$ then $f(a, b)$ is a saddle point of $f(x, y)$
- $D = 0$ the test is inconclusive so that $f(a, b)$ could be a local maximum, local minimum, a saddle point or none of the above!

EXAMPLE

Use the Second Derivative Test to classify the critical points of $f(x, y) = Ax^2 + Bxy + Cy^2$ based on the values of A, B and C.

GROUPWORK

McCallum, page 835, Example 6. Find the local maxima, local minima and saddle points of $f(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x$.

McCallum, page 836, Example 7. Classify the critical points of $f(x, y) = x^4 + y^4$, $g(x, y) = -x^4 - y^4$ and $h(x, y) = x^4 - y^4$.