Multivariable Calculus

Math 212 Spring 2015 — Fowler 309 MWF 9:35am - 10:30am

 — http://faculty.oxy.edu/ron/math/212/15/

Worksheet 11

TITLE The Chain Rule and Composition of Multivariable Functions
CURRENT READING McCallum, Section 14.6
HW #5 (DUE TUESDAY 02/24/15)
McCallum, *Section 14.3*: 3, 5, 6, 7, 14, 22.
McCallum, *Section 14.4*: 4, 7, 10, 18, 23, 36, 37, 50, 54*,55.
McCallum, *Section 14.5*: 7, 19, 25, 39, 52, 60, 66*.

SUMMARY
This worksheet discusses how to apply the Chain Rule in order to find the derivative of a function which is the composition of functions of multiple variables.

The Chain Rule For Single Variable Functions

RECALL: composition of single variable functions

Given a function $y = f(x)$ and another function $x = g(t)$ the composition of the function f with g, denoted $f \circ g$ or $y = f(g(t))$ is a new function which takes t as its input and y as its output.

RECALL: chain rule

Given a function $y = f(x)$ and another function $x = g(t)$, the derivative of a composition of functions $f \circ g$ or $f(g(t)$ is found using the **chain rule** and equals $f'(g(t))g'(t)$. Using the notation of Leibniz, the chain rule can be denoted:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{1}$$

Rate Of Change of A Composite Functions of Several Variables

Given a function $z = f(x, y)$ suppose that $x = g(t)$ and $y = h(t)$

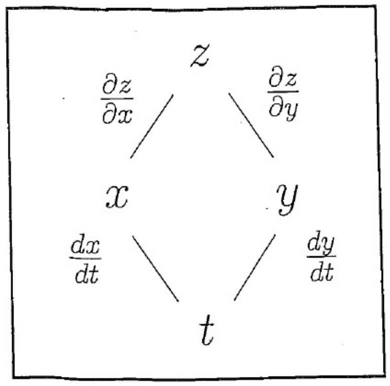
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{2}$$

EXAMPLE

McCallum, page 798, Example 2.
Consider the functions $g(x, y) = x\sin(y)$ where $x = t^2$ and $y = 2t + 1$.
Suppose $z = g(t)$.Compute $g'(t)$ directly and using the chain rule.

Visualizing The Chain Rule

Suppose z depends on x and y, both of which depend on t, we can draw a diagram depicting these relationships below:



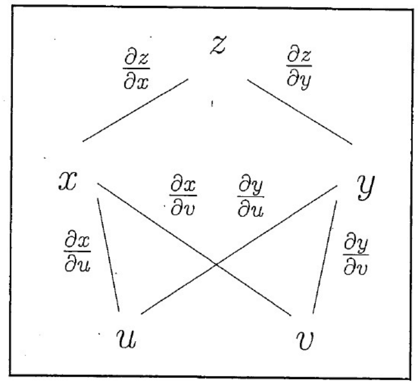
There are lines between z and both x and y and then since x and y are both functions of t there is a line from x to t and from y to t. Along each of these lines write down the appropriate derivative. By following all the possible paths from z to t and adding them up one obtains the following expression for $\frac{dz}{dt}$ where

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Exercise

Draw the diagram for $z = f(x, y)$ where $x = g(u, v)$ and $y = h(u, v)$.

Use the Chain Rule (assisted by the diagram) to write down expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$



Algorithm For Drawing The General Chain Rule Diagram

To find the rate of change of one variable with respect to another in a "chain" of differentiable functions which are compositions of each other:

1. Draw a diagram expression the relationship between the variables and label each link in the diagram with the derivative relating the variables at each end.
2. For each path between two variables, multiple together the derivatives for each step along the path.
3. Add the contributions from each path.

Exercise

Math 224 Spring 2003, Exam 1, Question 10. Draw a tree of variables for which this is the correct chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{dy}{du}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

EXAMPLE

McCallum, page 804, Exercise 29-30.

Let $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$ and $x(1, 2) = 5$, $y(1, 2) = 3$.
Given

$$\begin{array}{llll} f_x(1,2) = a & f_y(1,2) = c & x_u(1,2) = e & y_u(1,2) = p \\ f_x(5,3) = b & f_y(5,3) = d & x_v(1,2) = f & y_v(1,2) = q \end{array}$$

Calculate $z_u(1, 2)$ and $z_v(1, 2)$.

$\boxed{\text{GROUPWORK}}$

Suppose $w = x + y + z$, $x = u^2 + v^3, y = e^{uv}, z = u - v$ and $u = t + 1, v = e^t$. Our goal is to find $w'(0)$.

Use the Visual Chain Rule to assist you in calculating $w'(t)$ by drawing the appropriate chain rule diagram in the space below and then computing $w'(t)$ and $w'(0)$.

Find $w'(t)$.

Find $w'(0)$.