Worksheet 10

TITLE The Gradient Vector For Functions Of Three Variables
CURRENT READING McCallum, Section 14.5 (and 12.5)
HW #5 (DUE TUESDAY 02/24/15)
McCallum, *Section 14.3*: 3, 5, 6, 7, 14, 22.
McCallum, *Section 14.4*: 4, 7, 10, 18, 23, 36, 37, 50, 54*,55.
McCallum, *Section 14.5*: 7, 19, 25, 39, 52, 60, 66*.

SUMMARY
This worksheet discusses the gradient vector as applied to functions of three variables, i.e. $f(x, y, z)$. This will allow us to expand from our previous understanding of the gradient vector in $\mathbb{R}^2$.

RECALL: the gradient vector $\vec{\nabla} f$ in the plane

For surfaces $z = f(x, y)$ the gradient vector $\vec{\nabla} f = f_x\vec{i} + f_y\vec{j}$. The gradient vector has the following properties (when $\vec{\nabla} f \neq \vec{0}$):

- grad$f(a, b)$ points in the direction of maximum rate of change of $f(x, y)$
- grad$f(a, b)$ is a vector perpendicular to the contours of $f(x, y)$
- $||$grad$f(a, b)||$ is the maximum rate of change of $f(x, y)$ at the point (a, b)

The Gradient Vector in $\mathbb{R}^3$

Given a function $f(x, y, z)$ the gradient vector is computed at the point $a, b, c)$

$$\text{grad} f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k} \tag{1}$$

The Directional Derivative in $\mathbb{R}^3$

The directional derivative of a function $f(x, y, z)$ in the direction of
the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is

$$f_{\vec{u}}(a, b, c) = \text{grad} f(a, b, c) \cdot \hat{u}$$

Properties of the Gradient Vector In $\mathbb{R}^3$

Given that the function f is differentiable at the point (a, b, c) and grad $f(a, b, c) \neq \vec{0}$ the gradient vector in $\mathbb{R}^3$ has the following properties:

- grad $f(a, b, c)$ points in the direction of maximum rate of change of $f(x, y, z)$
- grad $f(a, b, c)$ is a vector perpendicular (i.e. normal) to the level surface of $f(x, y, z)$ at the point (a, b, c)
- $||$grad $f(a, b, c)||$ is the maximum rate of change of $f(x, y, z)$ at the point (a, b, c)

NOTE:

The properties of the gradient vector are the same in $\mathbb{R}^2$ or $\mathbb{R}^3$ (they just have slightly different interpretations depending on the vector space we are in).

`QUESTION:` What do the contours (i.e. level sets) of a function $f(x, y, z)$ look like?

EXAMPLE

McCallum, page 790, Example 2.
Consider the functions $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = -x^2 - y^2 - z^2$.
Calculate **and** interpret the meaning of the following vectors. What can we say about their direction?
(a) grad $f(0, 1, 1)$ **(b)** grad $f(1, 0, 1)$ **(c)** grad $g(0, 1, 1)$ **(d)** grad $g(1, 0, 1)$

Tangent Plane to the Level Surface of a Multivariable Function $f(x, y, z)$

A function $f(x, y, z)$ has level sets which are surfaces in $\mathbb{R}^3$. Since we can find the equation of the tangent plane to a level surface at a point (a, b, c) by knowing the normal vector to the surface at the point and the gradient vector by definition is perpendicular to the level surface at every point we can write down a formula for the **tangent plane to a level surface of** $f(x, y, z)$ **at** (a, b, c):

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

Exercise

McCallum, Page 792, Example 5.
Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, 2, 3)$.

GROUPWORK

McCallum, Page 795, Exercise 65. Two surfaces are said to be tangential at a point P if they have the same tangent plane at P. Show that $z = \sqrt{2x^2 + 2y^2 - 25}$ and $z = \frac{1}{5}(x^2 + y^2)$ are tangential at the point $(4, 3, 5)$.