
Multivariable Calculus

Math 212 Spring 2015
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Fowler 309 MWF 9:35am - 10:30am
<http://faculty.oxy.edu/ron/math/212/15/>

Class 1: Wednesday January 21

TITLE Introduction to Vectors (Notation and Terminology)

CURRENT READING McCallum, Section 13.1, 13.2

HW #1 (DUE WED 01/28/15)

McCallum, Section 13.1: 12, 23, 26, 31, 37; Section 13.2: 8, 26, 33, 43-48.

SUMMARY

One of the main concepts of this course will be the idea of a **vector**, which we will review in this class.

DEFINITION: **vector** and **scalar**

A **vector** is a quantity which has both magnitude and direction. A quantity which only has magnitude is called a **scalar**. We often denote vectors using the notation \vec{x} and some textbooks use \mathbf{x} to represent the vector x .

The magnitude (or length) of a vector \vec{x} is denoted $||\vec{x}||$.

GROUPWORK

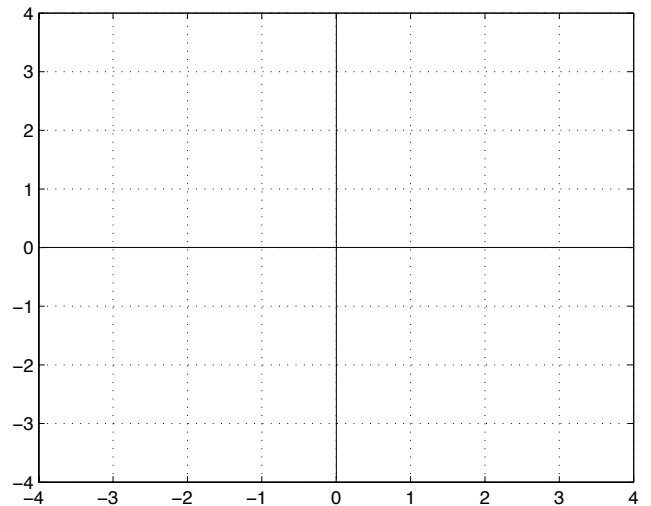
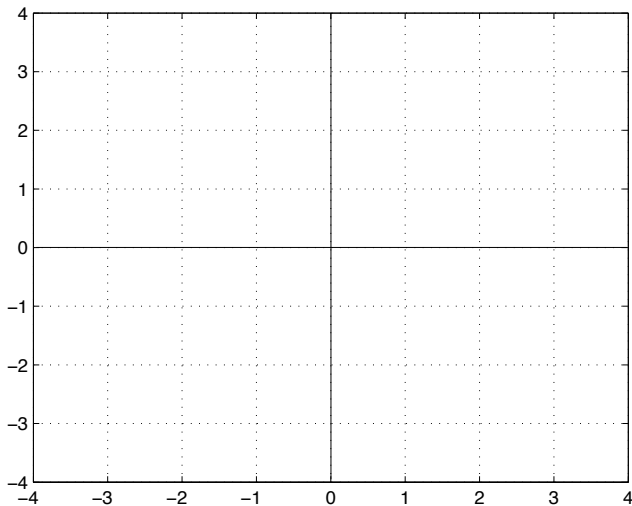
Write down the names of two physical quantities that are vectors and two that are scalars.

Points In Space

You may recall that in two-dimensions (i.e. \mathbb{R}^2) we denote points in space (i.e. the xy -plane) using ordered pairs (x, y) called co-ordinates. In three-dimensions (\mathbb{R}^3) this becomes the ordered triple (x, y, z) . Points in n -dimensional space (\mathbb{R}^n) are denoted by $(x_1, x_2, x_3, \dots, x_n)$ called n -tuples.

Displacement Vectors

The first type of vector your textbook discusses are **displacement vectors**. A displacement vector is a vector pointing from one point in space to another point in space. We denote \overrightarrow{AB} as the displacement vector from point A to point B . The displacement vectors which have the same magnitude and direction are the same regardless of where they are located.

**EXAMPLE**

Sketch the displacement vector \overrightarrow{AB} from the point A (2,1) to B (-1,-1) on the figure to the LEFT at the top of the page.

QUESTION: How many locations can one sketch \overrightarrow{AB} in the figure?

Position Vectors

We can also think of ordered pairs, ordered triples and n -tuples as **vectors**. These are sometimes called position vectors and represent the displacement from the origin of the coordinate system to the point in question.

Zero Vector

The **zero vector** is a vector with magnitude zero. It is also the position vector for the origin of a coordinate system.

Vectors in their Most General Form

In fact, in their most general or abstract form, a **vector** can just be thought of as a collection of n quantities or components. The general vector in this abstract form must obey some specific rules involving certain operations: **vector addition** and **scalar multiplication**.

Scalar Multiplication and Vector Addition**Exercise**

Consider the scalar $r = 1.5$ and position vectors $\vec{a} = (2, 2)$ and $\vec{b} = (-3, -2)$.

Evaluate the following expressions and sketch the position vectors on the figure to the RIGHT at the top of the page.

$$r\vec{a} = \underline{\hspace{4cm}}$$

$$\vec{a} + \vec{b} = \underline{\hspace{4cm}}$$

QUESTION: Do you notice any interesting geometric properties of these vector operations?

NOTE

- If λ is a scalar and \vec{v} is a vector, then $\lambda\vec{v}$ is parallel to \vec{v} , pointing in the same direction as \vec{v} if $\lambda > 0$ and in the opposite direction of $\lambda < 0$.
- The magnitude of $\lambda\vec{v}$ is $|\lambda|$ times the magnitude of \vec{v} . In other words, $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$

Vector Space

Vectors can also be defined more generally and theoretically as objects which belong to a **vector space**, which is defined below. For most of Math 212, we will just be thinking of a vector as an element of the vector space \mathbb{R}^n .

DEFINITION: Vector Space

A **vector space** is a set \mathcal{V} of objects (of *any* kind!) called **vectors**, with two operations, called **vector addition** and **scalar multiplication**, that satisfy the following ten properties:

- (1) \mathcal{V} is closed under vector addition: $\vec{v}, \vec{w} \in \mathcal{V} \Rightarrow (\vec{v} + \vec{w}) \in \mathcal{V}$
- (2) \mathcal{V} is closed under scalar multiplication: $c \in \mathbb{R}, \vec{v} \in \mathcal{V} \Rightarrow (c\vec{v}) \in \mathcal{V}$
- (3) Vector addition is commutative: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- (4) Vector addition is associative: $(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$
- (5) There is a unique additive identity in \mathcal{V} : $\vec{v} + \vec{0} = \vec{v}$
- (6) Each element in \mathcal{V} has a unique additive inverse: $\vec{v} + (-\vec{v}) = \vec{0}$
- (7) The scalar 1 acts as the multiplicative identity: $1\vec{v} = \vec{v}$
- (8) Scalar multiplication is associative: $c_1(c_2\vec{v}) = (c_1c_2)\vec{v}$.
- (9) Scalar multiplication is distributive w.r.t. vector addition: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.
- (10) Scalar multiplication is distributive w.r.t. scalar addition. $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$.

DEFINITION: closed under an operation

A set (e.g. vector space) is said to be “closed under an operation” when the result of the operation applied to any element of the set (e.g. vector), is also an element of the set.

GROUPWORK

Prove that \mathbb{R}^2 is a vector space, in other words that objects in \mathbb{R}^2 satisfy the ten properties that define a vector space. (**HINT**: what’s the most general notation for a vector in \mathbb{R}^2 ?)

Components of a Vector

Any displacement in \mathbb{R}^n can be expressed as a linear combination of displacements in the co-ordinate directions.

In \mathbb{R}^3 the coordinate directions are known as \vec{i} , \vec{j} and \vec{k} . (Some books depict these as \hat{i} , \hat{j} and \hat{k} .)

We say that we resolve the vector \vec{v} into its components by writing it as:

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

where $v_1\vec{i}$, $v_2\vec{j}$ and $v_3\vec{k}$ are the components of \vec{v} . This is often denoted as (v_1, v_2, v_3) or

sometimes as $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. (We will usually use the former notation in this class.)

Given a vector \vec{v} in \mathbb{R}^2 with length $\|\vec{v}\|$ pointing in the direction θ radians from the horizontal direction we can show using some trigonometry that

$$\vec{v} = (\|\vec{v}\| \cos \theta)\vec{i} + (\|\vec{v}\| \sin \theta)\vec{j}$$

Distance and Length

The **distance** between two points in \mathbb{R}^n is the **length** of the displacement vector representing the difference between the position vectors for the two points.

The **length** of a vector $\vec{x} = (x_1, x_2, x_3, \dots, x_n)$ in \mathbb{R}^n is denoted by $\|\vec{x}\|$ and computed by the formula $\sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$

Thus, given a point with position vector \vec{p} and another point with position vector \vec{q} the distance between them is $\|\vec{p} - \vec{q}\|$ (or, equivalently $\|\vec{q} - \vec{p}\|$)

Exercise

Find the magnitude of $(0, 4, -1, 3)$ and the distance between $(-1, 2, 1)$ and $(3, 0, 0)$.

Unit Vector

A unit vector is a vector whose magnitude is 1. Examples of unit vectors are the vectors \vec{i} , \vec{j} and \vec{k} . In order to “normalize” a non-unit vector \vec{v} by creating a unit vector \vec{u} (sometimes denoted as \hat{v}) pointing in the same direction as \vec{v} one simply has to multiply a vector by a scalar equal to the reciprocal of its magnitude, i.e.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

EXAMPLE

Let's find a unit vector that points in the direction of the vector $\vec{v} = \vec{i} + 3\vec{j}$