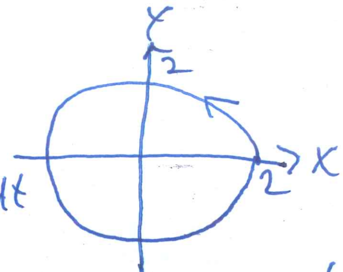
Consider the vector field $\vec{F}(x,y) = (x-y)\hat{\imath} + (x+y)\hat{\jmath}$ and the closed path C which is the circle of radius 2 centered at the origin traversed in the counter-clockwise direction. Evaluate the expression $\mathcal{I} = \oint_C \vec{F} \cdot d\vec{x}$ **two different** ways.

(a) *(5 points.)* Compute $\mathcal{I}$ directly.



$$I = \oint_C \vec{F} \cdot d\vec{x} = \int_0^{2\pi} \begin{pmatrix} 2\cos t - 2\sin t \\ 2\cos t + 2\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt$$

$x = 2\cos t$

$y = 2\sin t$

$0 \le t \le 2\pi$

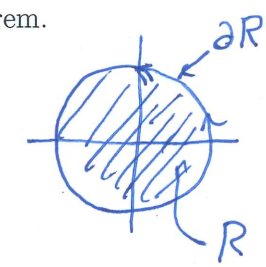
$$I = \int_0^{2\pi} \cancel{-4\cos t\sin t} + 4\sin^2 t + 4\cos^2 t + \cancel{4\sin t\cos t}\, dt$$

$$= \int_0^{2\pi} 4 \cdot 1\, dt = 4 \cdot 2\pi = \boxed{8\pi}$$

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix}$$

$$\vec{F}(\vec{x}(t)) = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = \begin{pmatrix} 2\cos t - 2\sin t \\ 2\cos t + 2\sin t \end{pmatrix}$$

(b) *(5 points.)* Compute $\mathcal{I}$ by evaluating a double integral and applying Green's Theorem.



$$\oint_{\partial R} \vec{F} \cdot d\vec{x} \overset{GT}{=} \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dA = \iint_R 2\, dA = I$$

$$\vec{F} = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$= 2 \cdot \text{Area of } R$$

$$= 2 \cdot (\pi \cdot 2^2)$$

$$= 2 \cdot 4\pi = \boxed{8\pi}$$

$$(\vec{\nabla} \times \vec{F}) \cdot \hat{k} = F_{2_x} - F_{1_y}$$

$$= 1 - (-1)$$

$$= 2$$