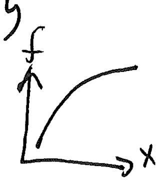
1. (*20 points.*) ANALYTIC, VISUAL, VERBAL. **Multiple Partial Derivatives, Clairault's Theorem, Contour Diagrams.**

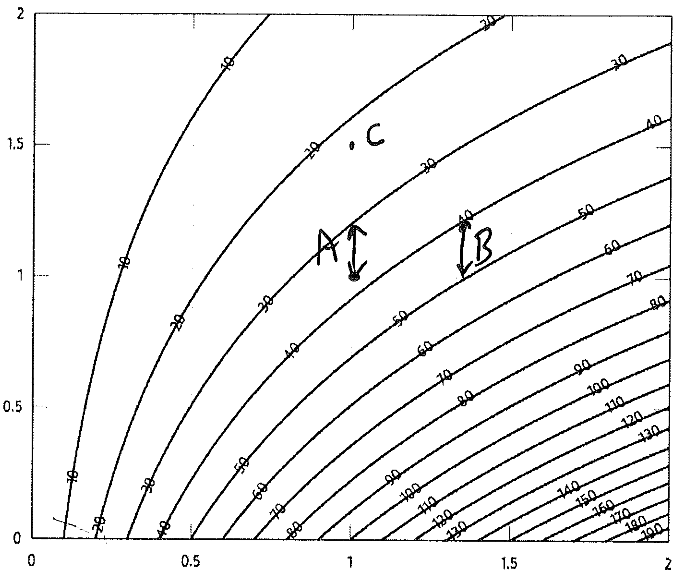
Consider the given contour diagram for a function $z = f(x,y)$ which is infinitely differentiable and everywhere continuous. The point of this problem is for you to estimate the sign (+, - or 0) of each of the partial derivatives given below.

$|f_x(A)| > |f_x(C)|$

$f_x(C)$ is less positive than $f_x(A)$

so $f_{xy} < 0$

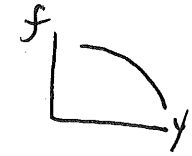




$|f_y(A)| < |f_y(B)|$

$f_y(B)$ is more negative than $f_y(A)$

$f_{yx} < 0$



1(a) (*8 points.*) What are the signs of $f_x(1,1)$ and $f_y(1,1)$? EXPLAIN YOUR ANSWERS!

$f_x > 0 \Leftrightarrow f \uparrow$ as $x \uparrow$

$f_y < 0 \Leftrightarrow f \downarrow$ as $y \uparrow$

The values of the function as represented by the numbers on the contour increase as one moves horizontally and they decrease as you move vertically

1(b) (*12 points.*) What are the signs of $f_{xx}(1,1)$, $f_{xy}(1,1)$, $f_{yx}(1,1)$ and $f_{yy}(1,1)$? EXPLAIN YOUR ANSWERS!

$f_{xy}(1,1) = f_{yx}(1,1) < 0$

$f_x \downarrow$ as $y \uparrow \Rightarrow f_{xy} < 0$

$f_y \downarrow$ as $x \uparrow \Rightarrow f_{xy} < 0$

$f_{xx}(1,1) > 0$ since $f_x \uparrow$ as $x \uparrow$

$f_{yy}(1,1) > 0$ since $f_y \uparrow$ as $y \uparrow$

2. (20 *points.*) VERBAL, ANALYTIC, COMPUTATIONAL. **Unconstrained Multivariable Optimization, Extreme Value Theorem, Repeated Partial Differentiation.**

2(a) (16 *points.*) Find the location of critical points of $f(x, y) = x^4 + y^4 - 4xy$ and use the Second Derivative Test to classify these points as local extrema or saddle points.

Solve $f_x = 0$ & $f_y = 0$

+4 $4x^3 - 4y = 0$ +1 $x^3 = y \Rightarrow x^9 = x$

+4 $4y^3 - 4x = 0$ $y^3 = x$

$x^9 - x = 0$

$x(x^8 - 1) = 0 \Rightarrow x = 0, 1, -1$

6 Critical points at $(0,0), (1,1), (-1,-1)$ +4

$f_{xx} = 12x^2$

+3 $f_{yy} = 12y^2$

$f_{xy} = -4 = f_{yx}$

$D = 144x^2y^2 - (4)^2$

$= 144x^2y^2 - 16$

6 $D(0,0) = -16 < 0 \Rightarrow$ SADDLE at $(0,0,0)$

$D(1,1) = 144 - 16 = 128 > 0$ with $f_{xx} > 0 \Rightarrow$ LOCAL MINIMUM at $(1,1,-2)$

$D(-1,-1) = 144 - 16 = 128 > 0$ with $f_{xx} > 0 \Rightarrow$ LOCAL MINIMUM at $(-1,-1,-2)$

2(b) (4 *points.*) Are the extrema you found in **2(a)** global extrema for $f(x, y)$? EXPLAIN YOUR ANSWER!

$(0,0,0)$ is a saddle so not an extrema.

$(-1,-1,-2)$ & $(1,1,-2)$ ~~clearly~~ ARE GLOBAL MIN because ~~[illegible]~~ there's no way to have $4xy > x^4 + y^4 + 2$ for $x,y \in [-1,1] \times [-1,1]$

There is no global max since $\lim_{\substack{x \to \pm\infty \\ y \text{ const}}} f(x,y) \to +\infty$ and $\lim_{\substack{x = y \\ y \to \pm\infty}} f(x,y) = +\infty$. Extreme value Theorem does not apply because domain is Not closed or bounded

3. (*20 points.*) ANALYTIC, VISUAL, VERBAL. **Constrained Multivariable Optimization, Lagrange Multipliers.**

Find the maximum and minimum temperatures T on the surface $x^2 + 6y^2 + 4z^2 = 49$ if the temperature at any point (x, y, z) in space is given by $T(x, y, z) = 2x - 6y + 3z$. [HINT: the numerical answers should be "nice numbers."]

Objective $T = 2x - 6y + 3z$ (2)

constraint $g = x^2 + 6y + 4z^2 - 49 = 0$

LME (10pts)

$$2 = T_x = \lambda g_x = \lambda 2x$$
$$-6 = T_y = \lambda g_y = \lambda 12y$$
$$3 = T_z = \lambda g_z = \lambda 8z$$
$$g = 0$$

+1

$$\lambda = \frac{2}{2x} = -\frac{6}{12y} = \frac{3}{8z}$$

$$\frac{1}{x} = \frac{-1}{2y} \Rightarrow y = -\frac{1}{2}x$$

$$\frac{1}{x} = \frac{3}{8z} \Rightarrow z = \frac{3}{8}x$$

$$x^2 + 6\left(-\frac{1}{2}x\right)^2 + 4\left(\frac{3}{8}x\right)^2 = 49$$

$$x^2 + \frac{6}{4}x^2 + 4\frac{9}{64}x^2 = 49$$

+6 x, y, z

$$x^2\left(1 + \frac{24}{16} + \frac{9}{16}\right) = 49$$

$$x^2\left(\frac{49}{16}\right) = 49$$

$$x^2 = 16$$

$$x = \pm 4$$

$$y = -\frac{1}{2}(\pm 4) = \mp 2$$

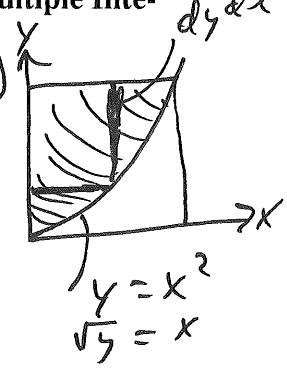
$$z = \frac{3}{8}(\pm 4) = \pm\frac{3}{2}$$

$T(4, -2, \frac{3}{2}) = 2\cdot 4 - 6\cdot -2 + 3\cdot\frac{3}{2} = 49$ ← GLOBAL MAX +4

$T(-4, 2, -3/2) = 2\cdot -4 - 6\cdot 2 + 3\cdot\frac{-3}{2} = -\frac{49}{2}$ ← GLOBAL MIN

4. (*30 points.*) ANALYTIC, VISUAL, COMPUTATIONAL. **Iterated Integration, Multiple Integration.**

3(a) (*10 points.*) Evaluate $\displaystyle\int_0^1 \int_{x^2}^1 \frac{x^3}{1+y^3}\, dy\, dx$ $= \displaystyle\int_0^1 \int_0^{\sqrt{y}} \frac{x^3}{1+y^3}\, dx\, dy$ (5pts) $dy\,dx$



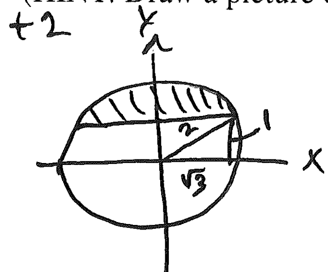
$= \frac{1}{4}\displaystyle\int_0^1 \frac{x^4}{1+y^3}\Big|_0^{\sqrt{y}} dy = \frac{1}{4}\int_0^1 \frac{y^2}{1+y^3}\, dy$ 2pts

$= \frac{1}{4}\cdot\frac{1}{3}\ln(1+y^3)\Big|_0^1$ 2pts

$= \frac{1}{12}\ln 2 - \frac{1}{3}\ln 1$

$= \frac{1}{12}\ln 2$ 2pts

3(b) (*10 points.*) Write down an integral which represents $\mathcal{A}$, the area of the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$ and the line $y = 1$. DO NOT EVALUATE THE INTEGRAL. (HINT: Draw a picture of the region of integration!)



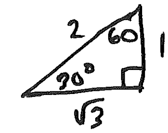
$\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-x^2}} dy\, dx$

$y = 1 \quad x^2 = 4 - y^2$

$= 4 - 1$

$x^2 = 3 \quad x = \pm\sqrt{3}$

3(c) (*10 points.*) Write down ANOTHER integral (different from your answer in **3(b)**) which also represents $\mathcal{A}$, the area of the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$ and the line $y = 1$. DO NOT EVALUATE THE INTEGRAL. (HINT: You can use Fubini's Theorem or Change Coordinate Systems!)



Fubini!

$\displaystyle\int_1^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\, dy$

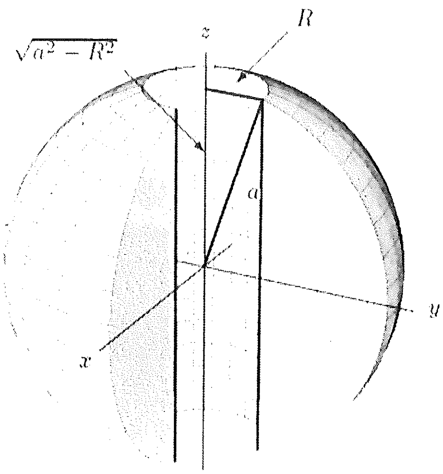
Polar Coordinates

$\displaystyle\int_{\pi/6}^{5\pi/6} \int_{1/\sin\theta}^{2} r\, dr\, d\theta$

$y = 1 = r\sin\theta$

$\frac{1}{\sin\theta} = r$

BONUS *[5 points total.]* ANALYTIC, VISUAL, COMPUTATIONAL. **Triple Integrals, Polar Coordinates, Cylindrical Coordinates.**



Show that the volume of the object (shown in the figure above) that remains when a cylindrical hole of radius R is bored through the center of a sphere of radius a (where $0 < R < a$) along the pole is equal to $\frac{4\pi}{3}(a^2 - R^2)^{3/2}$.

Cylindrical Coordinates +1

$$2\int_0^{2\pi}\int_R^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \qquad + 2$$

$z^2 + r^2 = a^2 \rightarrow$

$z = \pm\sqrt{a^2 - r^2}$

Spherical Coordinates

$$2\int_0^{2\pi}\int_{\sin^{-1}(r/a)}^{\pi - \sin^{-1}(r/a)}\int_{R/\sin\phi}^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$\rho^2 = x^2 + y^2 + z^2$

$x^2 + y^2 = R^2$

$\rho^2 = R^2 + z^2$

$\tan\alpha = \frac{R}{\sqrt{a^2 - r^2}}$

$\sin\alpha = \frac{R}{a}$

$$2\int_0^{2\pi}\int_R^a r\sqrt{a^2 - r^2}\, dr\, d\theta$$

$$-\frac{(a^2 - r^2)^{3/2} \cdot \frac{2}{3}}{+2}\Bigg|_{r=R}^{r=a} \cdot 2\pi \cdot 2$$

$=$

$0 + \frac{4\pi}{3}(a^2 - R^2)^{3/2}$

$z = \rho\cos\phi$

$\rho^2 = R^2 + \rho^2\cos^2\phi$

$\rho^2 \sin^2\phi = R^2$

$\rho = \frac{R}{\sin\phi}$

$=$