Exam 1: Multivariable Calculus

Math 212 Spring 2015 | Friday February 27
Prof. Ron Buckmire | 9:35am-10:30am

Name: BUCKMIRE

Directions:

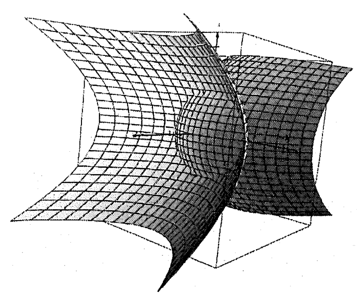
Read *all* problems first before answering any of them. This tests consists of three (3) problems (and a BONUS problem) on six (6) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, limited notes*, closed book, test. **No calculators or electronic devices may be used.**

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

***You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam when you hand it in.**
Questions Policy: FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!



No.	Score	Maximum
1		30
2		40
3		30
BONUS		5
Total		**100**

Academic Ethics: I, ______________________________, attest that I have followed all the rules displayed above to the letter and in spirit and this exam reflects my own work conducted according to Occidental's Spirit of Honor.

1. (*30 points.*) ANALYTIC, COMPUTATIONAL, VERBAL. **Equation of Lines, Vector Operations, Parametric Equations.**

Lines in $\mathbb{R}^3$ which are neither parallel nor have any points of intersection are called **skew lines**. Consider the equations of two skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{R}^3$, given in parametric and vector form, respectively:

$$\mathcal{L}_1 : x = 1 + t, \quad y = 1 + t, \quad z = t \qquad \text{and} \qquad \mathcal{L}_2 : \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

1(a) (*10 points.*) Show that the given lines are not **parallel** to each other. EXPLAIN YOUR ANSWER. (**HINT: vectors that are orthogonal to each other can not be parallel.**)

$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t$

$\vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

These lines are not parallel because $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is NOT a multiple of $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$.

Also $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = 1 - 3 + 2 = 0$, so they are orthogonal lines

1(b) (*10 points.*) Show that the given lines **do not intersect**, i.e. they have no points in common. SHOW ALL YOUR WORK.

$x = 1 + t = s$

$y = 1 + t = -3s$

$z = t = 1 + 2s$

Only way x and y are equal is if $t + 1 = 0 = s = -3s$

so $t = -1$ & $s = 0$

$-1 = 1 + 2 \cdot 0$

This is impossible

So there are no points of intersection

1(c) (*10 points.*) Write down equations representing three lines in $\mathbb{R}^3$ which are "skewed," i.e. they have no points in common with each other and are not parallel to each other. (**HINT: you may use $\mathcal{L}_1$ and $\mathcal{L}_2$ as two of your three skew lines if you wish but you do not have to.**) EXPLAIN HOW YOU KNOW YOUR THREE LINES ARE SKEWED.

Use cross product to find third vector orthogonal to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

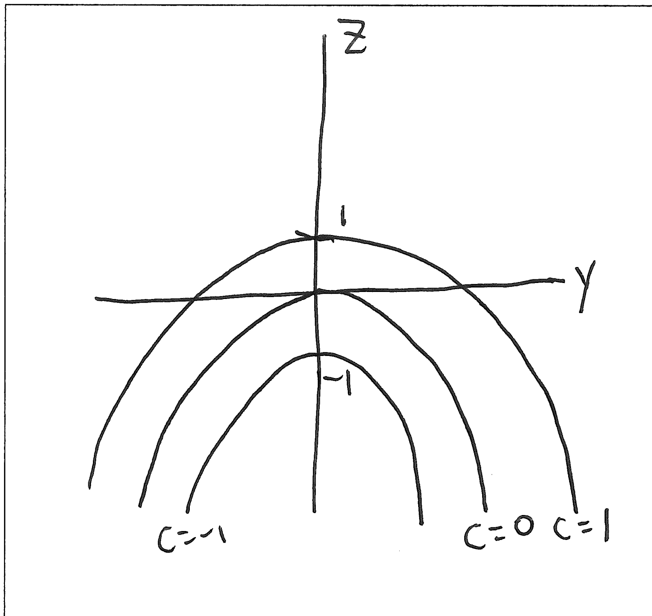
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -3 & 2 \end{vmatrix} = \hat{i}(2+3) - \hat{j}(2-1) + \hat{k}(-3-1) = 5\hat{i} - \hat{j} + 4\hat{k} = \begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix}$$

The origin $(0,0,0)$ is not on $\mathcal{L}_1$ or $\mathcal{L}_2$ so $\mathcal{L}_3 : \vec{x} = r \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

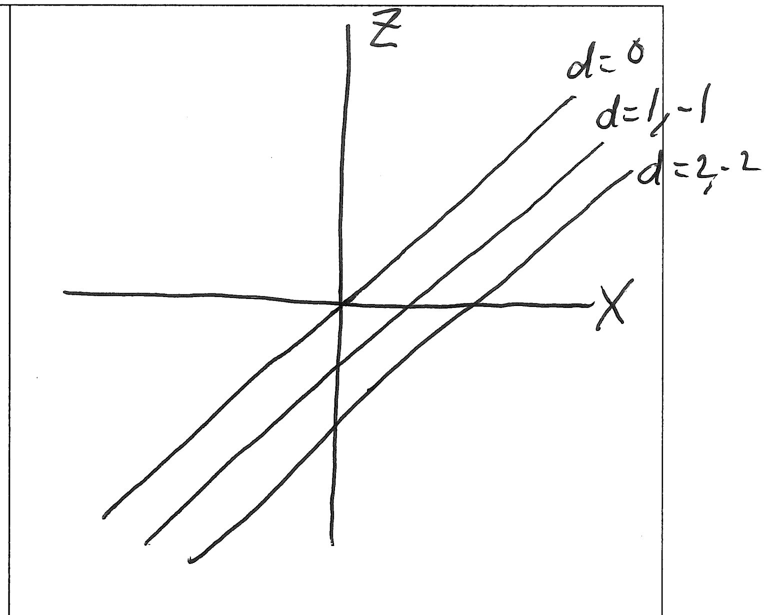
2. (*40 points.*) ANALYTIC, COMPUTATIONAL, VISUAL. **Multivariable Functions, Cross-Sections, Level Sets, Tangent Planes, Gradient Vector.**

Consider the surface $z = f(x, y) = x - y^2$ for all part of this question.

2(a) (*8 points.*) Sketch graphs of the x-intersection and y-intersections of this surface in the space below. LABEL YOUR AXES AND INDICATE WHICH CURVE REPRESENTS WHICH SLICE.



$z = f(x, y)$ when $x = c$ for $c = -1, 0$ or 1



$z = f(x, y)$ when $y = d$ for $d = -1, 0$ or 1

2(b) (*8 points.*) Compute $\vec{\nabla} f(3, -1)$.

$f_x = 1$ $\qquad \vec{\nabla} f(3,-1) = 1 \cdot \hat{i} + 2\hat{j} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$f_y = -2y$

$f_x(3,-1) = 1$

$f_y(3,-1) = -2(-1)$

$= 2$

2(c) (*8 points.*) Find the equation to the tangent plane to the surface $z = f(x, y) = x - y^2$ at the point $(3, -1)$. Make sure your equation is in the form $ax + by + cz + d = 0$. **HINT: What is normal vector to the surface at the point** $(3, -1)$**?**

$f(3,-1) = 3 - (-1)^2 = 3 - 1 = 2$

$z = f(3,-1) + f_x(3,-1)(x-3) + f_y(3,-1)(y - -1)$

$= 2 + 1 \cdot (x-3) + 2 \cdot (y+1)$

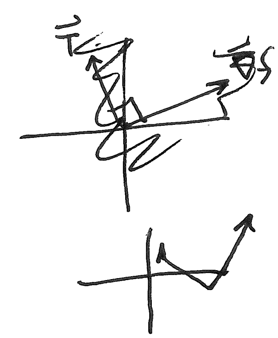
$= 2 + x - 3 + 2y + 2$

$z = x + 2y + 1 \implies \boxed{x + 2y - z + 1 = 0}$

2(d) (*8 points.*) Use your answer to **2(b)** to find the equation of the tangent line to the level set $f(x, y) = 2$ at the point $(3, -1)$. Make sure your tangent line equation is in the form $ax+by+c = 0$.
HINT: What is the slope of the tangent line to the level set $f(x, y) = 2$ at the point $(3, -1)$?

$x - y^2 = 2 \Rightarrow y^2 = x - 2$

At $(3,1)$ gradient vector points in $\begin{pmatrix}1\\2\end{pmatrix}$ direction so tangent points in $\begin{pmatrix}-2\\1\end{pmatrix}$ direction (i.e. orthogonal to gradient)



slope of tangent line is $-\frac{1}{2}$

because slope of normal is 2.

Use point-slope form

$$y - (-1) = -\frac{1}{2}(x-3)$$
$$y + 1 = -\frac{1}{2}(x-3)$$
$$-2(y+1) = x - 3$$
$$-2y - 2 = x - 3$$
$$-2y - 2 + 3 - x = 0 \Rightarrow -x - 2y + 1 = 0$$
$$x + 2y - 1 = 0$$

can also get slope by implicit diff

$1 - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

at $x = 3, y = -1$

$\frac{dy}{dx} = -\frac{1}{2}$

2(d) (*8 points.*) Recall $z = f(x, y) = x - y^2$. Sketch a graph of the level set $f(x, y) = 2$, the tangent line you found in part **2(c)** and the gradient vector you found in **2(b)** in the space below. LABEL YOUR FIGURE CAREFULLY!

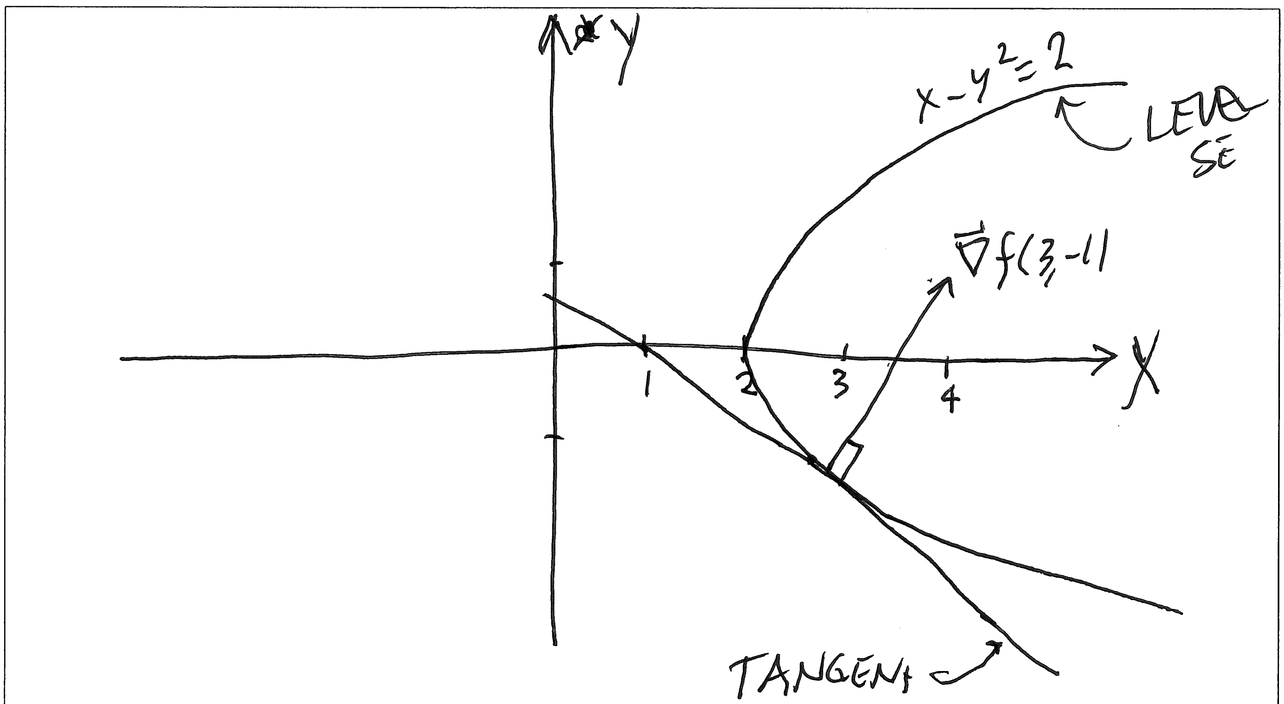
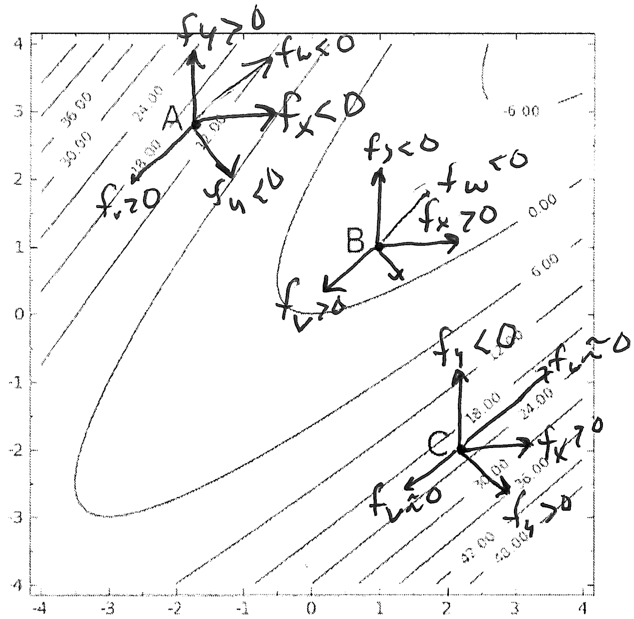


Figure contains level set $f(x, y) = 2$, tangent line to this level set at $(3, -1)$ and $\vec{\nabla} f(3, -1)$

3. (*30 points.*) ANALYTIC, VISUAL, VERBAL. **Partial Derivatives, Multivariable Functions, Contour Diagrams, Directional Derivative.** Consider the following figure which depicts the contour diagram of an unknown function $f(x, y)$. The goal of this problem is for you to demonstrate your understanding of the relative rate of change of an unknown function $f(x, y)$ in specific directions using only information from its contour diagram.

3(a) (*9 points.*) Indicate the relative sizes of the partial derivatives of $f(x, y)$ at the points A, B and C in the contour diagram by writing the appropriate partial derivative in the boxes below.



3(b) $\dfrac{\partial f}{\partial x}$

NEGATIVE	A	B	C	**POSITIVE**

3(c) $\dfrac{\partial f}{\partial y}$

NEGATIVE	C	B	A	**POSITIVE**

3(d) $f_{\vec{u}}$ where $\vec{u} = \hat{\imath} - \hat{\jmath}$, $\vec{v} = \hat{i} + j$

NEGATIVE	A	B	C	**POSITIVE**

3(b) (*7 points.*) Discuss the reasoning for your ordering you gave above in **3(a)** of the relative sizes of the partial derivative of f with respect to x at the points A, B and C, i.e your placement of $f_x(A)$, $f_x(B)$ and $f_x(C)$ in increasing order in the boxes.

$f_x(A) < 0$
$f_x(C) > 0$
$f_x(B) > 0$ but much smaller than $f_x(C)$ because contours are so far apart at B

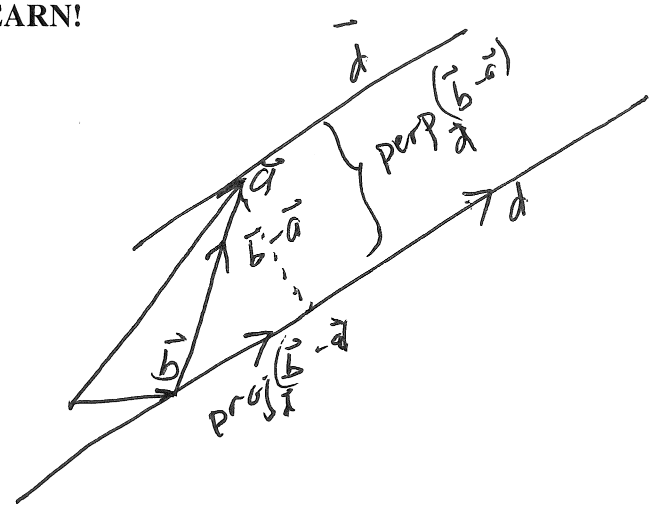
3(c) (*7 points.*) Discuss the reasoning for your ordering you gave above in **3(a)** of the relative sizes of the partial derivative of f with respect to y at the points A, B and C, i.e your placement of $f_y(A)$, $f_y(B)$ and $f_y(C)$ in increasing order in the boxes.

$f_y(A) > 0$
$f_y(B) < 0$
$f_y(C) < 0$ $\quad f_y(C) < f_y(B)$ because contours are closer together at C than B, so larger change

3(d) (*7 points.*) Discuss the reasoning for the ordering you gave above in **3(a)** of the relative sizes of the directional derivative of f in the direction $\vec{u} = \hat{\imath} - \hat{\jmath}$ at the points A, B and C, i.e your placement of $f_{\vec{u}}(A)$, $f_{\vec{u}}(B)$ and $f_{\vec{u}}(C)$ in increasing order in the boxes.

same as in (b) $f_{\vec{u}}(A) < 0$, $f_{\vec{u}}(B) > 0$, $f_{\vec{u}}(C) >> 0$

BONUS QUESTION. Vector Operations and Projection (*5 points.*)
Consider two parallel lines $\vec{x} = \vec{a} + \vec{d}t$ and $\vec{x} = \vec{b} + \vec{d}t$ in $\mathbb{R}^3$ where $\vec{a} \neq \vec{b}$. Obtain an expression for the minimum distance between these parallel lines. You may assume that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$. Explain in words how you would determine the distance between the two lines and then write down a mathematical expression involving the given parameters that would allow this distance to be calculated for a specific example. Assume that none of the given vectors $\vec{a}$, $\vec{b}$ or $\vec{d}$ are the zero vector.
EXPLAIN YOUR ANSWER THOROUGHLY AND SHOW ALL YOUR WORK, EXTRA CREDIT POINTS ARE HARD TO EARN!



MIN distance is the magnitude of $\text{perp}_{\vec d}(\vec b - \vec a)$

$= \|\text{perp}_{\vec d}(\vec b - \vec a)\|$

$= \|\vec b - \vec a - \text{proj}_{\vec d}(\vec b - \vec a)\|$

$= \left\|(\vec b - \vec a) - \left(\dfrac{(\vec b - \vec a)\cdot \vec d}{\vec d \cdot \vec d}\right)\vec d\right\|$

Find the vector between two points ($\vec a$ & $\vec b$) on the line, then the projection of that in the $\vec d$ direction, then subtract the result from $\vec b - \vec a$ to obtain perpendicular component. The length will be the magnitude of the difference.