

Worksheet 27

TITLE Divergence and Curl of a Vector Field
CURRENT READING McCallum, Section 19.3 and 20.1)
HW #12 (DUE Wednesday 12/1/14 5PM)
McCallum, *Section 18.4*: 1, 2, 3, 4, 15, 16, 20, 23*.
McCallum, *Chapter 18 Review*: 1, 2, 8, 15, 16, 17, 26, 45.
McCallum, *Section 19.3*: 1, 2, 3, 4, 6, 11, 27, 28.
McCallum, *Section 20.1*: 3, 4, 7, 13, 14, 28.

SUMMARY
This worksheet discusses the geometric and algebraic definitions of the curl and divergence of a vector field.

RECALL

Given a vector field $\vec{F}$ in $\mathbb{R}^2$ such that $\vec{F} = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$ the expression $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ is called the **scalar curl**.

DEFINITION: **scalar curl in** $\mathbb{R}^3$

Given a 3-D vector field with only two components $\vec{F}(x,y) = F_1(x,y)\hat{i} + F_2(x,y)\hat{j} + 0\hat{k}$ we can define the (badly-misnamed) **scalar curl** of $\vec{F}$ to be

$$\textbf{curl}\, \vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

NOTE: The curl of a 2-D vector field will either be pointed into the page (using the symbol $\otimes$) or out of the page (using the symbol $\odot$) or be the zero vector $\vec{0}$.

The Curl Of A Vector Field

DEFINITION: **vector curl** in $\mathbb{R}^3$

The **curl** of a vector field $\vec{F}(\vec{x})$ in $\mathbb{R}^3$ is a vector property denoted by **curl** $\vec{F}(\vec{x})$ and defined as $\vec{\nabla} \times \vec{F}$ where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ in $\mathbb{R}^3$ and $\vec{\nabla}$ is the vector operator $\dfrac{\partial}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}\hat{j} + \dfrac{\partial}{\partial z}\hat{k}$.

$$\textbf{curl}\, \vec{F} = \vec{\nabla} \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Algebraically, you could think of the curl as the following determinant:

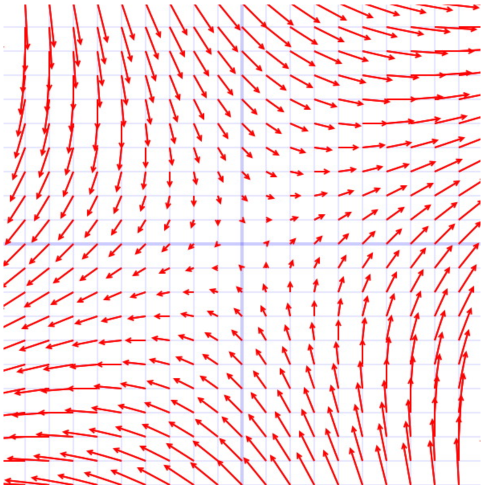
$$\textbf{curl}\, \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

NOTE: The curl is the **cross product** of the gradient operator with a vector field $\vec{F}$, so it is a vector quantity.

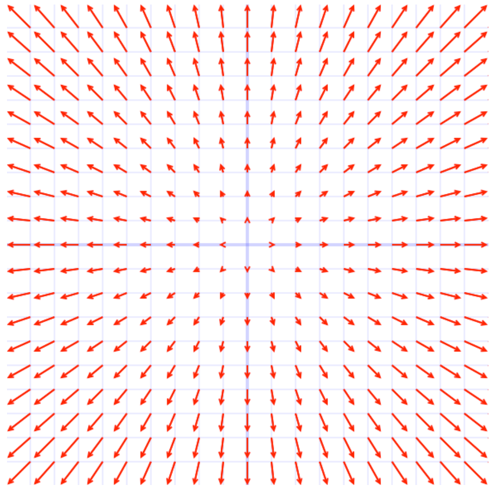
The Curl Of Some Of Our Favorite Vector Fields

EXAMPLE

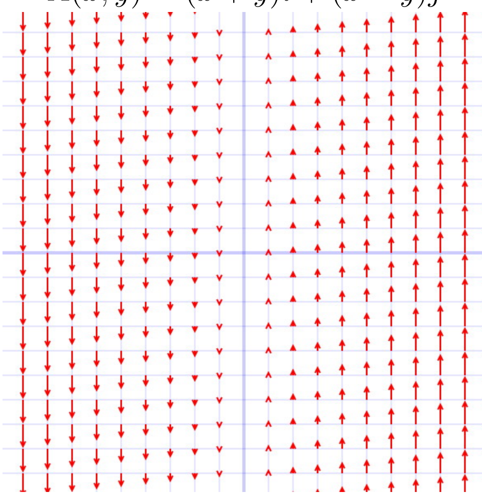
Let's find the curl of some of our favorite planar vector fields.



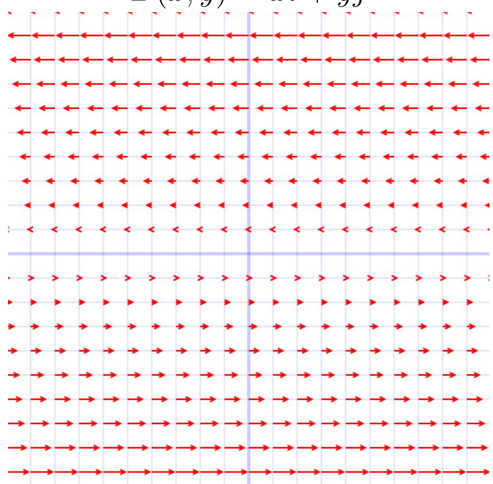
$\vec{A}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath}$



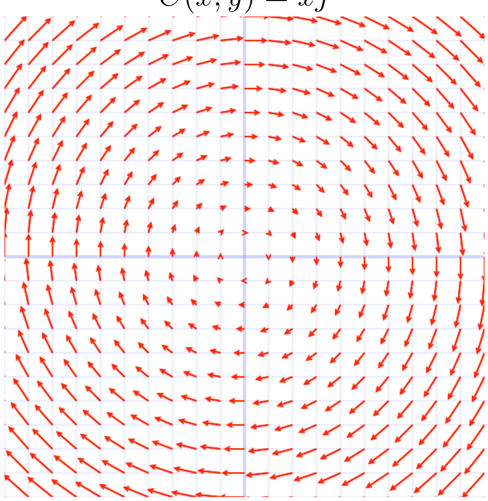
$\vec{B}(x,y) = x\hat{\imath} + y\hat{\jmath}$



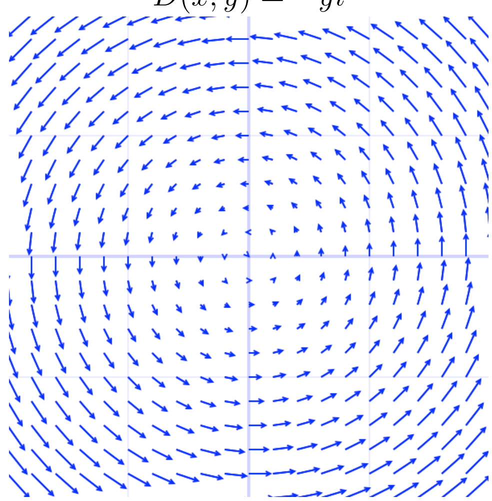
$\vec{C}(x,y) = x\hat{\jmath}$



$\vec{D}(x,y) = -y\hat{\imath}$



$\vec{E}(x,y) = y\hat{\imath} - x\hat{\jmath}$



$\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$

Geometric Understanding Of Curl

DEFINITION: circulation density

The **circulation density** of a smooth vector field $\vec{F}$ around the direction of a unit vector $\hat{n}$ is defined, provided the limit exists, to be

$$\text{circ}_{\hat{n}}\vec{F} = \lim_{\text{Area}\to 0} \frac{\displaystyle\oint_C \vec{F}\cdot d\vec{x}}{\text{Area inside } C} = \lim_{\text{Area}\to 0} \frac{\text{Circulation of } \vec{F} \text{ around } C}{\text{Area inside } C} = (\vec{\nabla}\times\vec{F})\cdot\hat{n}$$

where C is a closed curve in a plane perpendicular to $\hat{n}$ positively oriented using the right-hand rule. (When the Right-Hand Thumb points in direction of $\hat{n}$ Other Fingers are curled in direction of traversal around C.)

CONCEPTUAL UNDERSTANDING OF CURL

The direction the vector **curl** of a vector field $\vec{F}$ points in is the direction for which the circulation density of $\vec{F}$ is the GREATEST.

The magnitude of the vector **curl** of a vector field $\vec{F}$ is the circulation density of $\vec{F}$ around the direction $\vec{\nabla}\times\vec{F}$ points in.

If the circulation density is zero around *every* direction then we say the curl is $\vec{0}$ and describe such a vector field as **irrotational**.

Recall that gradient fields have the property that every line integral around a closed path is zero so this means that **all gradient fields are irrotational**, which can be expressed mathematically as

$$\vec{\nabla}\times\vec{\nabla}\phi = \vec{0} \text{ for any potential function } \phi$$

Divergence of a Vector Field

DEFINITION: divergence

The **divergence** of a vector field $\vec{F}(\vec{x})$ is a **scalar** property denoted by $\mathbf{div}\vec{F}(\vec{x})$ defined as the trace of the Jacobian matrix, i.e. the sum of the diagonal elements of this matrix. In particular, if one considers $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ in $\mathbb{R}^3$ where $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ then the divergence of $\vec{F}$ can be defined as

$$\mathbf{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

NOTE: The divergence is the **dot product** of the gradient operator with a vector field $\vec{F}$, so it is a scalar quantity.

GROUPWORK

Find the Divergence of the six vector fields depicted earlier on this worksheet.

Properties Of Gradient, Divergence and Curl As Vector Calculus Operations

The divergence $\vec{\nabla}\cdot\Box$ and curl $\vec{\nabla}\times\Box$ can be thought of as differential operations that are applied to vector fields, and produce scalars and vectors, respectively. The gradient operator $\vec{\nabla}\Box$ is applied to a scalar function and outputs a vector field. Given scalar functions ϕ and ψ and vector fields $\vec{F}$ and $\vec{G}$ the following properties apply to the gradient, curl and divergence operators.

Distributivity

$$\begin{aligned}\vec{\nabla}\cdot(\vec{F}+\vec{G}) &= \vec{\nabla}\cdot\vec{F}+\vec{\nabla}\cdot\vec{G}\\ \vec{\nabla}\times(\vec{F}+\vec{G}) &= \vec{\nabla}\times\vec{F}+\vec{\nabla}\times\vec{G}\\ \vec{\nabla}(\phi+\psi) &= \vec{\nabla}\phi+\vec{\nabla}\psi\end{aligned}$$

Product Rules

$$\begin{aligned}\vec{\nabla}\cdot(\phi\vec{F}) &= \vec{F}\cdot(\vec{\nabla}\phi)+\phi\vec{\nabla}\cdot\vec{F}\\ \vec{\nabla}\times(\phi\vec{F}) &= \phi(\vec{\nabla}\times\vec{F})+(\vec{\nabla}\phi)\times\vec{F}\\ \vec{\nabla}(\phi\psi) &= \psi(\vec{\nabla}\phi)+\phi(\vec{\nabla}\psi)\end{aligned}$$

Repeated Applications ("Second Derivatives")

$$\begin{aligned}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) &= \textbf{div curl }\vec{F}=0\\ \vec{\nabla}\times(\vec{\nabla}\phi) &= \textbf{curl grad }\phi=\vec{0}\\ \vec{\nabla}\cdot(\vec{\nabla}\phi) &= \nabla^2\phi=\Delta\phi=\textbf{div grad }\phi\\ \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{F})-\ \nabla^2\vec{F}=\textbf{curl curl }\vec{F}\end{aligned}$$

Exercise

How many possible binary arrangements of **div**, **grad** and **curl** are there?

QUESTION: How many of these are well-defined operations? How many of these are identically zero?

EXAMPLE

For $\vec{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ and $\vec{\nabla}=\dfrac{\partial}{\partial x}\hat{i}+\dfrac{\partial}{\partial y}\hat{j}+\dfrac{\partial}{\partial z}\hat{k}$ let's confirm the vector calculus identities **div curl** $\vec{F}=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0$ and $\vec{\nabla}\times(\vec{\nabla}\phi)=$ **curl grad** $\phi=\vec{0}$.