Multivariable Calculus

Math 212 §2 Fall 2014 — Fowler 309 MWF 11:45am - 12:40pm
 — http://faculty.oxy.edu/ron/math/212/14/

Worksheet 26

TITLE Path-Dependent Vector Fields and Green's Theorem
CURRENT READING McCallum, Section 18.4
HW #12 (DUE Wednesday 11/19/14 5PM)
McCallum, *Section 18.1*: 6, 11,12,13,14, 22, 27.
McCallum, *Section 18.2*: 4, 5, 6, 7, 8, 20, 33..
McCallum, *Section 18.3*: 3, 4, 5, 6, 18, 21, 30.

SUMMARY
This worksheet discusses how to identify path-dependent vector fields, introduces the **curl** operator and provides Green's Theorem as a technique for evaluating line integrals in such fields.

RECALL

The Fundamental Theorem of Line Integrals says that if C is **ANY** path from the point $\vec{x}_A$ to $\vec{x}_B$

$$\int_C \vec{\nabla} f \cdot d\vec{x} = f(\vec{x}_B) - f(\vec{x}_A)$$

QUESTION: What do you think would happen to the value of $\int_C \vec{\nabla} f \cdot d\vec{x}$ if $\vec{x}_A = \vec{x}_B$, i.e. C was a closed curve?

DEFINITION: **circulation**

The circulation of a vector field $\vec{F}$ around a closed curve C is defined to be the value of the line integral $\int_C \vec{F} \cdot d\vec{x}$, which is sometimes written as $\oint_C \vec{F} \cdot d\vec{x}$.

THEOREM

A vector field is a path-independent field if and only if $\int_C \vec{F} \cdot d\vec{x} = 0$ for every closed curve C lying in the domain of $\vec{F}$. In other words, if

$$\text{Circulation of } \vec{F} \text{ around } C = 0 \iff \vec{F} \text{ is a path-independent vector field}$$

$$\oint_C \vec{F} \cdot d\vec{x} = 0 \iff \vec{F} = \vec{\nabla}\phi$$

CONCEPTUAL UNDERSTANDING

The idea that the line integral of a conservative vector field over any closed curve (i.e. the circulation in a gradient field) must be equal to zero follows directly from the Fundamental Theorem for Line Integrals by putting the beginning and ending point to the same point.

Algebraic Condition To Show A 2D Vector Field Is NOT A Gradient Field

THEOREM

The vector field $\vec{F} = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ is a gradient field if and only if $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$

As always, the **contrapositive** statement of the theorem is also true, i.e.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \neq 0 \Longleftrightarrow \vec{F} \text{ is NOT a gradient vector field}$$

EXAMPLE

McCallum, page 986, Example 3. Show that the vector field $\vec{F}(x,y) = 2xy\hat{\imath} + xy\hat{\jmath}$ is path-dependent. (HINT: show that $\vec{F}$ is NOT a gradient field.)

DEFINITION: the **scalar curl** in $\mathbb{R}^2$

Given a vector field $\vec{F}$ in $\mathbb{R}^2$ such that $\vec{F} = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ the expression $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ is called the **scalar curl**.

GROUPWORK

Consider the path C from $(1,0)$ to $(-1,0)$ that goes through the point $(0,1)$ following two linear paths. Compute the following line integrals after first determining which are path-dependent and which path-independent. DO YOU NOTICE ANY PATTERNS?

(a) $\int_C 2xy\, dx + x^2\, dy$

(b) $\int_C \vec{F} \cdot d\vec{x}$ where $\vec{F} = ye^{xy}\hat{\imath} + xe^{xy}\hat{\jmath}$

(c) $\int_C \vec{F} \cdot d\vec{x}$ where $\vec{F} = x^{2/3}\hat{\imath} + e^{7y}\hat{\jmath}$

(d) $\int_C -y\, dx + x\, dy$

NOTE We sometimes write the line integral $\int_C \vec{F} \cdot d\vec{x}$ where $\vec{F} = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ as the integral $\int_C F_1\, dx + F_2\, dy$.

Green's Theorem

THEOREM

Suppose C is a piecewise smooth simple closed curve that is the boundary of a region $\mathcal{R}$ in the plane oriented so that the region is on the left as we traverse C. Given a smooth vector field $\vec{F} = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$ defined on an open region containing $\mathcal{R}$ and C, then

$$\oint_C \vec{F} \cdot d\vec{x} = \iint_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \tag{1}$$

CONCEPTUAL UNDERSTANDING

Green's Theorem allows us to swap a line integral in $\mathbb{R}^2$ for an area integral and vice-versa. It says that the line integral of a closed path is equal to the integral of the scalar curl over the area enclosed by the closed path.

NOTE

Suppose the $\vec{F}$ in Green's Theorem is a gradient field? Then since we know that the circulation must be zero (from the Fundamental Theorem of Line Integrals) AND we also know that the scalar curl is going to be zero (since we know that gradient fields have scalar curl always equal to zero). Thus the left hand and right hand sides of the expression will be equal.

EXAMPLE

McCallum, page 993, Exercise 13. Let $\vec{F} = xy\hat{j}$. Use Green's Theorem to calculate the circulation of $\vec{F}$ around the curve C, oriented counter-clockwise. C is the square $0 \le x \le 1,\ 0 \le y \le 1$.

Exercise

McCallum, page 993, Exercise 12. Let $\vec{F} = -y\hat{i} + x\hat{j}$. Use Green's Theorem to calculate the circulation of $\vec{F}$ around the curve, oriented counter-clockwise. C is the unit circle centered at the origin.

Application of Green's Theorem: Using Line Integrals To Compute Area!
One cool thing about Green's Theorem is that it allows you to find the areas of shapes that might be difficult to compute otherwise. All you have to do is pick a vector field which has scalar curl equal to 1.

Let $\vec{F} = \frac{-y}{2}\hat{i} + \frac{x}{2}\hat{j}$. We can show that the scalar curl of $\vec{F}$ is exactly equal to 1. This means that using Green's Theorem, the line integral of $\vec{F}$ around a closed curve C will be exactly equal to the area enclosed by C.

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{x} &= \iint_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\, dA \\
\oint_C F_1\, dx + F_2\, dy &= \iint_{\mathcal{R}} \left(\frac{1}{2} - \frac{1}{2}\right)\, dA \\
\oint_C \frac{-y}{2}\, dx + \frac{x}{2}\, dy &= \iint_{\mathcal{R}} 1\, dA \\
\frac{1}{2}\oint_C x\, dy - y\, dx &= \text{Area of } \mathcal{R}
\end{aligned}$$

EXAMPLE

We can use the above result to show that the area of the ellipse is equal to πab by evaluating the line integral of $\vec{F} = \frac{-y}{2}\hat{i} + \frac{x}{2}\hat{j}$ along the path given by $\vec{x} = a\cos t\hat{i} + b\sin t\hat{j}$ for $0 \le t \le 2\pi$.

Exercise

Show that the vector field $\vec{F} = x\hat{j}$ and an appropriate of application of Green's Theorem can confirm that the area of a circle of radius R is πR^2.

GROUPWORK

McCallum, page 994, Exercise 24. Calculate $\oint_C (x^2 - y)\,dx + (y^2 + x)\,dy$ if C is

(a) the circle $(x-5)^2 + (y-4)^2 = 9$ oriented counter clockwise

(b) the circle $(x-a)^2 + (y-b)^2 = R^2$ oriented counter clockwise