Multivariable Calculus

Math 212 §2 Fall 2014 — Fowler 309 MWF 11:45am - 12:40pm

 — http://faculty.oxy.edu/ron/math/212/14/

Worksheet 11

TITLE Second Order Partial Derivatives
CURRENT READING McCallum, Section 14.7
HW #6 (DUE WEDNESDAY 10/08/14)
McCallum, *Section 14.6*: 4, 11, 12, 26, 34, 35, 47*.
McCallum, *Section 14.7*: 6, 7, 8, 12, 19, 24, 30, 31,41*.
McCallum, *Section 14.8*: 3, 12, 19*.
McCallum, *Chapter 14*: 2, 14, 35, 45, 64*

SUMMARY
This worksheet discusses higher order partial derivatives of multivariable functions and introduces the concept of the mixed partial derivative.

RECALL: second derivative

Given an infinitely differentiable function $y = f(x)$ its derivative $f'(x)$ represents the slope of the graph of the function at any point and $f''(x)$ represents the concavity of the graph. Also, $f'(x)$ represents the instantaneous rate of change of $f(x)$ at a point while $f''(x)$ represents the instantaneous rate of change of $f'(x)$.

The Second-Order Partial Derivatives of $f(x, y)$

DEFINITION: f_{xx}, f_{xy}, f_{yy} and f_{yx}

Given a function $z = f(x, y)$ with continuous partial derivatives we can not only find the rate of change with f with respect to x and the rate of change of f with respect to y but the rate of change of *those functions* with respect to x and y also!

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx} \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy} \text{ and } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

These expressions above are referred to as the second order partial derivatives of $f(x, y)$

EXAMPLE

McCallum, page 812, Exercise 4.
Compute the four second-order partial derivatives of $f(x, y) = e^{2xy}$

`QUESTION:` Do you notice a relationship between f_{xy} and f_{yx}?

When Mixed Partial Derivatives Are Equal

THEOREM

(**Clairault's Theorem**) If f_{yx} and f_{xy} are continuous at some point (a, b) found in a disc $(x-a)^2 + (y-b)^2 \leq D$ for some $D > 0$ on which $f(x, y)$ is defined, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Applications of the Second-Order Partial Derivatives

Recall (from *Worksheet #8*) that the local linearization of a function $f(x, y)$ near the point (a, b) is given by the tangent plane

$$f(x, y) \approx P(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \tag{1}$$

Taylor Polynomial Approximations

Note that the expression on the right hand side of (1) can be thought of as Taylor Polynomial of Degree 1 approximating $f(x, y)$ near (a, b) for a function that has continuous first-order partial derivatives.

We can expand this idea from (1) to improve our approximation of this function. If $f(x, y)$ has continuous second-order partial derivatives we can produce a Taylor Polynomial of Degree 2 approximating $f(x, y)$ near (a, b):

$$\begin{aligned} f(x, y) \approx Q(x, y) = \quad & f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{f_{xx}(a, b)}{2}(x-a)^2 \\ & + f_{xy}(a, b)(x-a)(y-b) + \frac{f_{yy}(a, b)}{2}(y-b)^2 \end{aligned}$$

Exercise

McCallum, page 811, Example 5.

Find the Taylor Polynomial of degree 2 at the point (1,2) for the function $f(x, y) = \dfrac{1}{xy}$.

GROUPWORK

You are told that there is a function f whose partial derivative $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Do you believe this? PROVE YOUR ANSWER!

The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that $\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$.

The gas law for fixed mass m of an ideal gas at the absolute temperature T, pressure P and volume V is $PV = mRT$ where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$