Multivariable Calculus

Math 212 §2 Fall 2014 | Fowler 309 MWF 11:45am - 12:40pm

 | http://faculty.oxy.edu/ron/math/212/14/

Worksheet 8

TITLE The Tangent Plane, Differentials and Linear Approximations
CURRENT READING McCallum, Section 14.3
HW #4 (DUE WED 09/24/14)
McCallum, *Section 12.6*: 24, 28, 35, 40, 52.
Section 14.1: 10, 11, 12, 13, 17, 18, 25,26, 36, 37, 41, 48, 49.
Section 14.2: 8, 9, 14, 24, 25, 30, 34, 36, 39, 51, 52, 65*.

SUMMARY
This worksheet discusses the multivariable analogue of the linear approximation to a single-variable function, often visualized using tangent lines, to use tangent planes as linear approximations to surfaces $z = f(x, y)$. We will also introduce the concept of infinitesimal differentials.

RECALL: tangent line approximation for $f(x)$ at (a,$f(a)$)

For single variable functions $f(x)$, we can approximate the graph of the function $y = f(x)$ with its tangent line $y = T(x)$ at $x = a$ given by

$$y = T(x) = f(a) + f'(a)(x - a)$$

Of course, you should also recognize the Tangent Line approximation as the First-Degree Taylor Polynomial Approximation of $f(x)$ at $x = a$.

RECALL: local linearity

A function $y = f(x)$ is said to be locally linear at a point $x = a$ if as one zooms in to that point it can be approximated more and more accurately by its tangent line at that point. Local linearity of a function is a proxy (i.e. conceptual stand-in) for **differentiability** of the function at that point.

QUESTION: Can you write down an example of a function $f(x)$ which is locally linear at the origin? How about a function that is not locally linear at the origin?

Tangent Plane to the Surface of a Multivariable Function

DEFINITION: **tangent plane**

Given that a surface $z = f(x, y)$ has continuous **partial derivatives** $f_x(a, b)$ and $f_y(a, b)$ then the equation of the Tangent plane at (a, b) is given by

$$z = P(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (1)$$

EXAMPLE

McCallum, Page 772, Example 1. Find the equation of the tangent plane at the surface $z = x^2+y^2$ at the point $(3, 4)$.

Multivariable Functions Can Be Locally Linear Also

The **local linearization** of a surface $z = f(x, y)$ at appoint (a, b) is when the surface can be approximated by the tangent plane to the surface at that point.
In other words,

$$f(x, y) \approx P(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

This approximation is called a **locally linearization** of the surface even though we are using a tangent plane because a plane is linear in the two variables x and y.

EXAMPLE

McCallum, Page 773, Example 2. Find the local linearization of $F(x, y) = x^2 + y^2$ at the point $(3, 4)$ and use it to estimate the value of $f(2.9, 4.2)$ and $f(2, 2)$.

The Differential

Recall that in single variable calculus one can relate the change in output, Δy or Δf, of a function $y = f(x)$ to a change in input Δx at any point x_0 using the expression

$$\Delta y \approx f'(x_0)\Delta x \qquad \text{or } \Delta f \approx f'(x_0)\Delta x \tag{2}$$

There is an equivalent analogue (2) to in Multivariable Calculus for the change in the output of a surface $z = f(x, y)$ at the point (x_0, y_0) compared to the changes of each input variable Δx and Δy.

$$\Delta z \text{ or } \Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

DEFINITION: **differential of** $z = f(x, y)$

The **differential** df (or dz), at a point (a, b) is the linear function of dx and dy given by the equation:

$$df = f_x(a, b)dx + f_y(a, b)dy$$

or in general (i.e. at any point) the differential of $f(x, y)$ can be written as

$$df = f_x\, dx + f_y\, dy \tag{3}$$

The differential can be thought of as a really, really small (i.e. infinitesimal) value associated with the variable that appears after the d.

GROUPWORK

McCallum, page 777, Exercise #23. Find the differential of $f(x, y) = \sqrt{x^2 + y^3}$ at the point (1,2). Use it to estimate the value of $f(1.04, 1.98)$.

QUESTION: What's the difference (in meaning and in value) between Δz and dz?