Multivariable Calculus

Math 212 §2 Fall 2014 | Fowler 309 MWF 11:45am - 12:40pm

 | http://faculty.oxy.edu/ron/math/212/14/

Class 2: Friday August 29

TITLE The Dot Product and Vector Equations Of Lines and Planes
CURRENT READING McCallum, Section 13.3
HW #1 (DUE WED 09/03/14)
McCallum, Section 13.1: 12, 23, 26, 31, 37; Section 13.2: 8, 26, 33, 43-48.

SUMMARY
One of the most important operations associated with vectors is the **dot product**. We shall introduce that, concept and also discuss how to write equations that represent geometric objects like lines and planes using vectors and the idea of orthogonality.

Dot Product

The dot product is a very useful operation on vectors that can be defined in two different, but equivalent ways.

Dot Product: Geometric Definition
The dot product of two vectors $\vec{w}$ and $\vec{w}$ can be defined in terms of the magnitudes $||\vec{v}||$ and $||\vec{w}||$ and the angle θ between the two vectors where θ is in radians in the range between 0 and π.

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \, ||\vec{w}|| \cos\theta$$

Dot Product: Algebraic Definition
Given two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ the dot product can be computed algebraically:

$$\vec{x} \cdot \vec{y} = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n$$

NOTE

- The dot product takes as input two vectors but outputs a scalar quantity.
- The dot product includes the product of two magnitudes and the factor $\cos\theta$ with $0 \leq \theta \leq \pi$.

QUESTION **What is the range of possible values of the dot product of two vectors?**

Properties of the Dot Poduct

For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$
Positivity: $\vec{u} \cdot \vec{u} > 0$, (except when $\vec{u} = \vec{0}$)
Symmetry: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
Additivity: $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
Associativity: $(r\vec{u}) \cdot \vec{v} = r(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (r\vec{v})$, $\forall r \in \mathbb{R}$

Exercise

Let $\vec{a} = \vec{i}$ and $\vec{b} = 2\vec{i} + 2\vec{j}$. Compute $\vec{a} \cdot \vec{b}$ geometrically and algebraically.

NOTE If we have a vector $\vec{v} \in \mathbb{R}^3$ we can use the dot product to find the components in the coordinate directions of $\vec{v} = (v_1, v_2, v_3)$: $v_1 = \vec{v} \cdot \vec{i}, \quad v_2 = \vec{v} \cdot \vec{j}, \quad v_3 = \vec{v} \cdot \vec{k}$.

Orthogonality

When the dot product is zero this means that the two vectors are orthogonal (i.e. perpendicular)

$$\vec{v} \cdot \vec{w} = 0 \Longleftrightarrow \text{ Angle between } \vec{v} \text{ and } \vec{w} \text{ is 90 degrees}$$

NOTE: $\vec{j} \cdot \vec{k} = 0$, $\vec{i} \cdot \vec{j} = 0$, $\vec{i} \cdot \vec{k} = 0$. But $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$ and $\vec{k} \cdot \vec{k} = 1$.

Magnitude of a Vector

The dot product can be used to compute the magnitude of a vector easily:

$$||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}} \Longleftrightarrow ||\vec{x}||^2 = \vec{x} \cdot \vec{x}$$

GROUPWORK

Determine which, if, any, of the following vectors are perpendicular to one another.
$\vec{u} = \vec{i} + \sqrt{3}\vec{k}, \quad \vec{v} = \vec{i} + \sqrt{3}\vec{j}, \quad \vec{w} = \sqrt{3}\vec{i} + \vec{j} - \vec{k}$

QUESTION **Without using the dot product, how can you tell which vector has the largest magnitude?**

Application of the Dot Product: Work Done

The work W done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is $W = \vec{F} \cdot \vec{d}$.

Application of the Dot Product: Equations of Lines and Planes

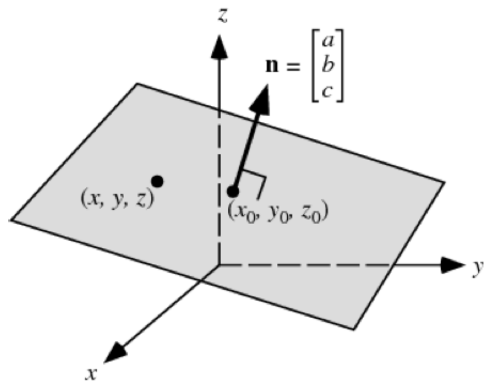
One key application of the Dot Product is finding the equation of a plane. If one thinks carefully about what a plane in $\mathbb{R}^3$ looks like it should be very clear that the best way to define a plane is the set of all points which are orthogonal to the plane's **normal vector**. And we know we can use the dot product to determine when two vectors are orthogonal to each other!

Normal Vector of a Plane in $\mathbb{R}^3$

The **normal vector** of a plane is the vector which is perpendicular to *every* displacement vector formed between any two points that lie in that plane.

Equations of a Plane in $\mathbb{R}^3$

The main way we often think of planes in Euclidean space (i.e. the space we are used to living in where lines are perfectly "straight" and go on forever) is to define a plane in $\mathbb{R}^3$ as a **two-dimensional geometric object** consisting of the infinite set of points that have the property that the displacement vector between any two points in the plane is orthogonal to the plane's normal vector. The figure below demonstrates a plane with a specific point with position vector $\vec{p} = (x_0, y_0, z_0)$ and normal vector $\vec{n} = (a, b, c)$.



General Form of the Equation of a Plane in $\mathbb{R}^3$

In other words, given a plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and the point P_0 on the plane with coordinates (x_0, y_0, z_0) the equation of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1}$$

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

By letting $d = ax_0 + by_0 + cz$ the above equation (1) can be re-written as

$$ax + by + cz = d \tag{2}$$

This equation (2) is called the **general form of the equation of a plane**.

NOTE: the coefficients of x, y and z in the general equation of the plane always make up the components of the normal vector to the plane, i.e. $\vec{n} = (a, b, c)$.

Normal Form of the Equation of a Plane in $\mathbb{R}^3$

However, if you look more closely at (1) we can see that if you consider any point P in $\mathbb{R}^3$ with position vector $\vec{p} = (x, y, z)$ and the position vector $\vec{p}_0$ of the specific point P_0 in the plane as (x_0, y_0, z_0) then one could re-write this equation (1) using the dot product as

$$(\vec{p} - \vec{p}_0) \cdot \vec{n} = 0 \tag{3}$$

But this equation (3) can also be written more simply as

$$\vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{p}_0 \tag{4}$$

This last result (4) is known as the **normal form of the equation of a plane**.

EXAMPLE

Find normal vectors to the following planes: (a) $x - y + 2z = 5$ (b) $z = 5x + y$

GROUPWORK

McCacllum, pg. 737, Example 5.
(a) Find the equation of the plane perpendicular to $\vec{n} = -\vec{i} + 3\vec{j} + 2\vec{k}$ passing through the point $(1, 0, 4)$.
(b) Find a vector parallel to the plane.

Equations of a Line in $\mathbb{R}^2$ and $\mathbb{R}^3$

The main way we often think of lines in Euclidean space is to define a line in $\mathbb{R}^n$ as the set of points composing the **one dimensional geometrical object** connecting two distinct points in space.

General Form of the Equation of a Line in $\mathbb{R}^2$
The **general form** of the equation of a line L in $\mathbb{R}^2$ is

$$ax + by = c \tag{5}$$

In this case the vector $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a **normal vector** to the direction represented by line L in $\mathbb{R}^2$.

Normal Form of the Equation of a Line in $\mathbb{R}^2$
The **normal form** of the equation of a line L in $\mathbb{R}^2$ is

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \text{ or } \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p} \tag{6}$$

In the case of Equation 6 the non-zero vector $\vec{n}$ is again a **normal vector** to the line L and $\vec{p}$ is a particular given point on the line L. Notice that Equation (6) representing a **line in** $\mathbb{R}^2$ has exactly the same form as the equations (3) and (4) which represent **a plane in** $\mathbb{R}^3$!

The difference is what space the vectors in question ($\mathbb{R}^2$ versus $\mathbb{R}^3$) one is referring to. What equations 6,(3) and (4) have in common is that they represent equations for $(n-1)$-dimensional geometric objects living in $\mathbb{R}^n$.

Vector Form of the Equation of a Line in $\mathbb{R}^n$
The **vector form** of the equation of a line L in $\mathbb{R}^2$ (or $\mathbb{R}^n$) is $\vec{x} = \vec{p} + t\vec{d}$. In this case the non-zero vector $\vec{d}$ is a **direction vector** for the line L, $\vec{p}$ is the position vector for a particular given point on the line L and $\vec{x}$ is the position vector for any point on the line.