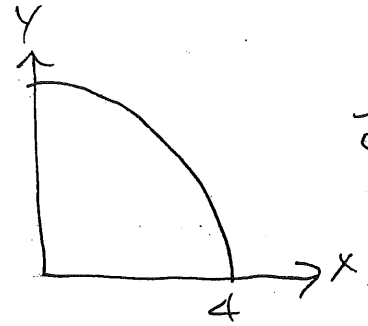
4. (*20 points.*) **Multiple Integration.** SPRING 2004

a. (*10 points*) Evaluate $\displaystyle\int\!\!\int_R ye^x\,dA$ where R is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region R).



$$\int_0^{\pi/2}\int_0^4 r\sin\theta\, e^{r\cos\theta}\, r\,dr\,d\theta = \int_0^4\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta\,dr$$

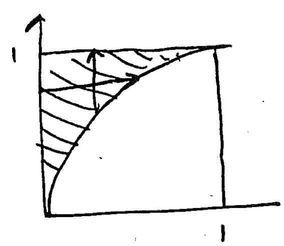
$$\int_0^4 -re^{r\cos\theta}\Big|_0^{\pi/2} dr = \int_0^4 -re^{r\cdot 0} + re^r\,dr$$

$$= -\frac{r^2}{2}\Big|_0^4 + re^r - e^r\Big|_0^4$$

$$= -8 + 4e^4 - e^4 - (0 - e^0)$$

$$= \boxed{-7 + 3e^4}$$

b. (*10 points*) Consider $\displaystyle\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} dz\,dy\,dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.



$$\int_0^1\int_0^{y^2}\int_0^{1-y} dz\,dx\,dy = \int_0^1\int_0^{y^2} 1-y\,dx\,dy$$

$$= \int_0^1 y^2(1-y)\,dy = \int_0^1 -y^3+y^2\,dy = -\frac{y^4}{4}+\frac{y^3}{3}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

3. (*20 points.*) **Iterated Integration.** SPRING 2004

a. (*10 points*) Evaluate $\int_{-3}^{0}\int_{0}^{2}\int_{-1}^{1} \cos(x+y+z) - xyz\, dx\, dz\, dy$

$$\int_{-3}^{0}\int_{0}^{2} \sin(x+y+z) - \frac{x^2}{2}yz \Big|_{-1}^{1} dz\,dy = \int_{-3}^{0}\int_{0}^{2} \sin(1+y+z) - \sin(-1+y+z) - \left(\tfrac{1}{2}yz - \tfrac{1}{2}yz\right) dz\,dy$$

$$= \int_{-3}^{0} -\cos(1+y+z) + \cos(-1+y+z) \Big|_{0}^{2} dy$$

$$= \int_{-3}^{0} -\cos(3+y) + \cos(1+y) + \cos(1+y) - \cos(-1+y)\, dy$$

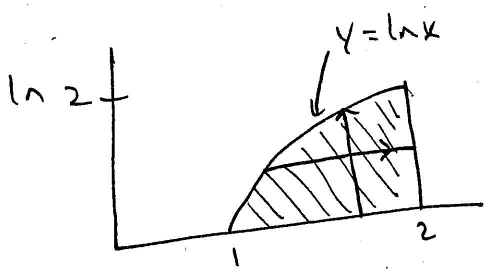
$$= -\sin(3+y) + 2\sin(1+y) - \sin(-1+y) \Big|_{-3}^{0}$$

$$= -\sin 3 + 2\sin 1 - \sin(-1) - \left[-\sin 0 + 2\sin(-2) - \sin(-4)\right]$$

$$= -\sin 3 + 2\sin 1 + \sin 1 - 2\sin(-2) + \sin(-4)$$

$$= \boxed{3\sin 1 + 2\sin 2 - \sin 3 - \sin 4}$$

b. (*10 points*) Evaluate $\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x}\, dy\, dx$ $= \int_{0}^{\ln 2}\int_{e^y}^{2} \frac{1}{x}\, dx\, dy$



$$= \int_{0}^{\ln 2} \ln x \Big|_{e^y}^{2} dy = \int_{0}^{\ln 2} \ln 2 - \ln(e^y)\, dy$$

$$= \int_{0}^{\ln 2} \ln 2 - y\, dy = (\ln 2)^2 - \frac{(\ln 2)^2}{2}$$

$$= \boxed{\frac{(\ln 2)^2}{2}}$$

$$\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x}\, dy\, dx = \int_{1}^{2} \frac{\ln x}{x}\, dx$$

$$= \frac{(\ln x)^2}{2} \Big|_{1}^{2}$$

$$= \frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{(\ln 2)^2}{2}$$

SPRING 2004

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
The "geometric mean" of n numbers is defined as $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots x_n = c$, where c is a constant.

a. (*10 points*) Find the maximum value of the geometric mean of n positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider f^n instead of f!]

$$\phi = f^n = x_1 x_2 x_3 x_4 \ldots x_n$$
$$g = x_1 + x_2 + x_3 + \ldots + x_n$$

$$\vec{\nabla}\phi = \vec{\nabla}(f^n) = \lambda \vec{\nabla} g$$

$$\nabla(f^n) = (x_2 x_3 x_4 \ldots x_n, x_1 x_3 x_4 \ldots x_n, x_1 x_2 x_4 \ldots x_n, \ldots, x_1 x_2 \ldots x_{n-1})$$

$$\vec{\nabla} g = (1, 1, 1, 1, \ldots, 1)$$

$$x_2 x_3 \ldots x_n = \lambda$$
$$x_1 x_3 \ldots x_n = \lambda$$
$$x_1 x_2 x_4 \ldots x_n = \lambda$$
$$\vdots$$
$$x_1 x_2 \ldots x_{n-1} = \lambda$$
$$x_1 + x_2 + \ldots x_n = c$$

$$\prod_{i=1}^{n} x_i = \lambda x_1$$
$$\prod_{i=1}^{n} x_i = \lambda x_2$$
$$\prod_{i=1}^{n} x_i = \lambda x_3$$
$$\vdots$$
$$\prod_{i=1}^{n} x_i = \lambda x_n$$

~~$\lambda(x_1 + x_2 + \ldots x_n)$~~

$\lambda x_1 = \lambda x_2 = \lambda x_3 = \ldots = \lambda x_n$

either $\lambda = 0$

or

$x = x_1 = x_2 = x_3 = x_n$

If $\lambda = 0$ one $x_i = 0$ (impossible)

$nx = c \Rightarrow x = \frac{c}{n}$

$\phi = \sqrt[n]{\left(\frac{c}{n}\right)^n} = \frac{c}{n} \leftarrow$ MAX

b. (*10 points*) You can deduce from part (a) that the geometric mean of n numbers is always less than or equal to the arithmetic mean, that is:

$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same n numbers?

when $x_1 = x_2 = x_3 = \ldots = x_n = X$

geometric mean $= \sqrt[n]{X^n} = X$ ✓

arithmetic mean $= \frac{nX}{n} = X$ ✓

When $\sum_{i=1}^{n} x_i = c$ the max value of $\sqrt[n]{\prod_{i=1}^{n} x_i} = \frac{c}{n}$

so $\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n}$

EXTRA CREDIT (*10 points.*) **Unconstrained Multivariable Optimization**
Consider $f(x, y) = x^4 + y^4 - 4xy + 1$.

SPRING 2009

a. (*5 points*) Find the three critical points of $f(x, y)$.

Critical Points occur at $\vec{\nabla} f = \vec{0}$

$f_x = 4x^3 - 4y = 0$ $\quad y = x^3$

$f_y = 4y^3 - 4x = 0$ $\quad x = y^3$

$y = y^9$

$y^9 - y = 0$

$(y^8 - 1)y = 0$

$(y^4 - 1)(y^4 + 1)y = 0$

$(y^2 - 1)(y^2 + 1)(y^4 + 1)y = 0$

$y = 0 \quad y = 1, y = -1$

$x = y^3 = 0 \quad x = 1, \quad x = -1$

$(0, 0, 1)$

$(1, 1, -1)$

$(-1, -1, -1)$

b. (*5 points*) Use the Second Derivative Test to classify each of the three critical points of $f(x, y)$.

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

At $(0, 0, 1)$ $\quad D = 0 \cdot 0 - (-4)^2 = -16 < 0 \Rightarrow$ SADDLE

At $(1, 1, -1)$ $\quad D = 12 \cdot 12 - (-4)^2 = 144 - 16 = 128 > 0$ LOCAL MIN

$f_{xx} > 0 \quad f_{yy} > 0$

At $(-1, -1, -1)$ $\quad D = 12 \cdot 12 - (-4)^2$

$= 144 - 16 > 0$ LOCAL MIN

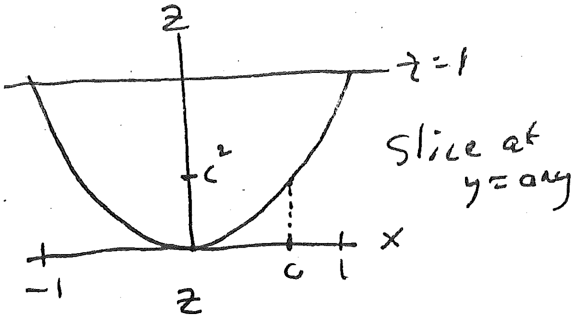
$12 = f_{xx} > 0$ & $f_{yy} > 0$

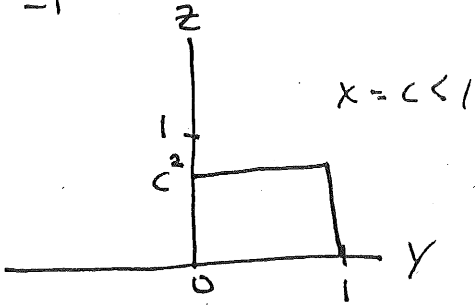
FALL 2005

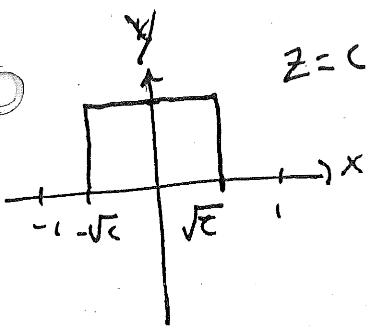
4. (*20 points.*) **Iterated Integration.**

Consider the iterated integral for $V = \int_{-1}^{1}\int_{0}^{1}\int_{x^2}^{1} dz\ dy\ dx = \frac{4}{3}$

(a) (*12 points.*) Write down 3 (**three**) of the 5 (**five**) other possible triple iterated integrals which represent the exact same value V. **HINT: There is no dependence of z upon y)** DO NOT EVALUATE THESE INTEGRALS.







1st simplest, switch y and x!

$$\int_0^1\int_{-1}^1\int_{x^2}^1 dz\,dx\,dy$$

$$=\int_0^1\int_0^1\int_{-\sqrt z}^{\sqrt z} dx\,dz\,dy$$

$$=\int_0^1\int_0^1\int_{-\sqrt z}^{\sqrt z} dx\,dy\,dz$$

$$=\int_{-1}^1\int_{x^2}^1\int_0^1 \cancel{dz\,dx\,dy}\ dy\,dz\,dx$$

$$=\int_0^1\int_{-\sqrt z}^{\sqrt z}\int_0^1 dy\,dx\,dz$$

(b) (*8 points.*) Use any one of the iterated integrals you wrote down in part (a) to confirm the value of V.

$$\int_0^1\int_{-1}^1 1-x^2\,dx\,dy = \int_0^1 x-\frac{x^3}{3}\Big|_{-1}^{1} dy = \int_0^1\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)dy$$

$$=\int_0^1 \frac{4}{3}\,dy = \frac{4}{3}\ \checkmark$$

$$\int_0^1\int_0^1\int_{-\sqrt z}^{\sqrt z} dx\,dz\,dy = \int_0^1\int_0^1 2\sqrt{z}\,dz\,dy = \int_0^1 2\cdot\frac{2}{3}z^{3/2}\Big|_0^1 dy$$

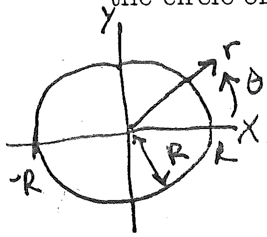
$$=\int_0^1 \frac{4}{3}\,dy = \frac{4}{3}\ \checkmark$$

FALL 2005

2. (*20 points.*) **Multiple Integration.**

The goal of this question is to evaluate $\int_0^\infty e^{-x^2}\,dx = \lim_{a\to\infty}\int_0^a e^{-x^2}\,dx$.

(a) (*10 points.*) Find $I(R) = \iint_{D_R} e^{-(x^2+y^2)}\,dx\,dy$ when D_R is $x^2+y^2 \le R^2$ (the interior of the circle of radius R centered at the origin). **HINT: pick a useful coordinate system!**



$$\int_0^{2\pi}\int_0^R e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \frac{e^{-r^2}}{-2}\Big|_0^R\,d\theta = \int_0^{2\pi} \frac{e^{-R^2}-1}{-2}\,d\theta$$

$dxdy \to r\,dr\,d\theta$

$$= 2\pi \cdot \frac{e^{-R^2}-1}{-2}$$

$$= \pi\left(1-e^{-R^2}\right)$$

(b) (*5 points.*) Take your answer $I(R)$ to **(b)** and then let $R \to \infty$. What is $\lim_{R\to\infty}\iint_{D_R} e^{-(x^2+y^2)}dx\ \ dy$?

$$\lim_{R\to\infty} I(R) = \pi\left(1-e^{-R^2}\right) = \pi \qquad \left(\text{since } \lim_{R\to\infty} e^{-R^2} = 0\right)$$

(c) (*5 points.*) Given that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2$ then what is the value of $\int_0^\infty e^{-x^2}\,dx$?

$$\pi = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2 \Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

since e^{-x^2} is symmetric about $x=0$

SPRING 2006

(f) Use the Method of Lagrange Multipliers to obtain the minimum value of $f(x, y) = x^2 + xy + y^2$ on the constraint set $g(x, y) = x + y - 2 = 0$.

$$\left.\begin{aligned}\vec{\nabla} f &= \lambda \vec{\nabla} g \\ g &= 0\end{aligned}\right\}$$

$f_x = 2x + y \quad f_{xx} = 2$
$f_y = x + 2y \quad f_{yy} = 2$
$g_x = 1 \quad f_{xy} = 1$
$g_y = 1$

$2x + y = \lambda$
$2y + x = \lambda$
$x + y = 2$

$D = f_{xx} f_{yy} - f_{xy}^2$
$= 2 \cdot 2 - 1^2 = 3 > 0$

$2x + y = 2y + x$
$2x - x = 2y - y$
$x = y$

$f_{xx} > 0$

$x + x = 2$
$2x = 2$
$x = 1 = y$

$f(1,1) = 1^2 + 1^2 + 1^2 = 3$

so $(1, 1, 3)$ is a minimum on constraint

$x + y = 2$ is not closed & bounded

There is no max

(g) How would the maximum and minimum on the constraint set change if the constraint set $g(x, y)$ were changed to $h(x, y) = x^2 + y^2 - 4$? Find the extrema of $f(x, y) = x^2 + xy + y^2$ subject to the constraint $h(x, y) = 0$ and EXPLAIN YOUR ANSWER.

$\vec{\nabla} f = \lambda \vec{\nabla} h$
$h = 0$

$f_x = 2x + y$
$f_y = 2y + x$
$h_x = 2x$
$h_y = 2y$

$2x + y = \lambda 2x$
$2y + x = \lambda 2y$
$x^2 + y^2 = 4$

$2xy + y^2 = \lambda 2xy$
$2yx + x^2 = \lambda 2yx$
$2xy + y^2 = 2yx + x^2$
$y^2 = x^2 \Rightarrow y = \pm x$
$x^2 + x^2 = 4$
$x^2 = 2$
$x = \pm\sqrt{2}$
$y = \pm\sqrt{2}$

$x^2 + y^2 = 4$ is closed & bounded so global max/min is found

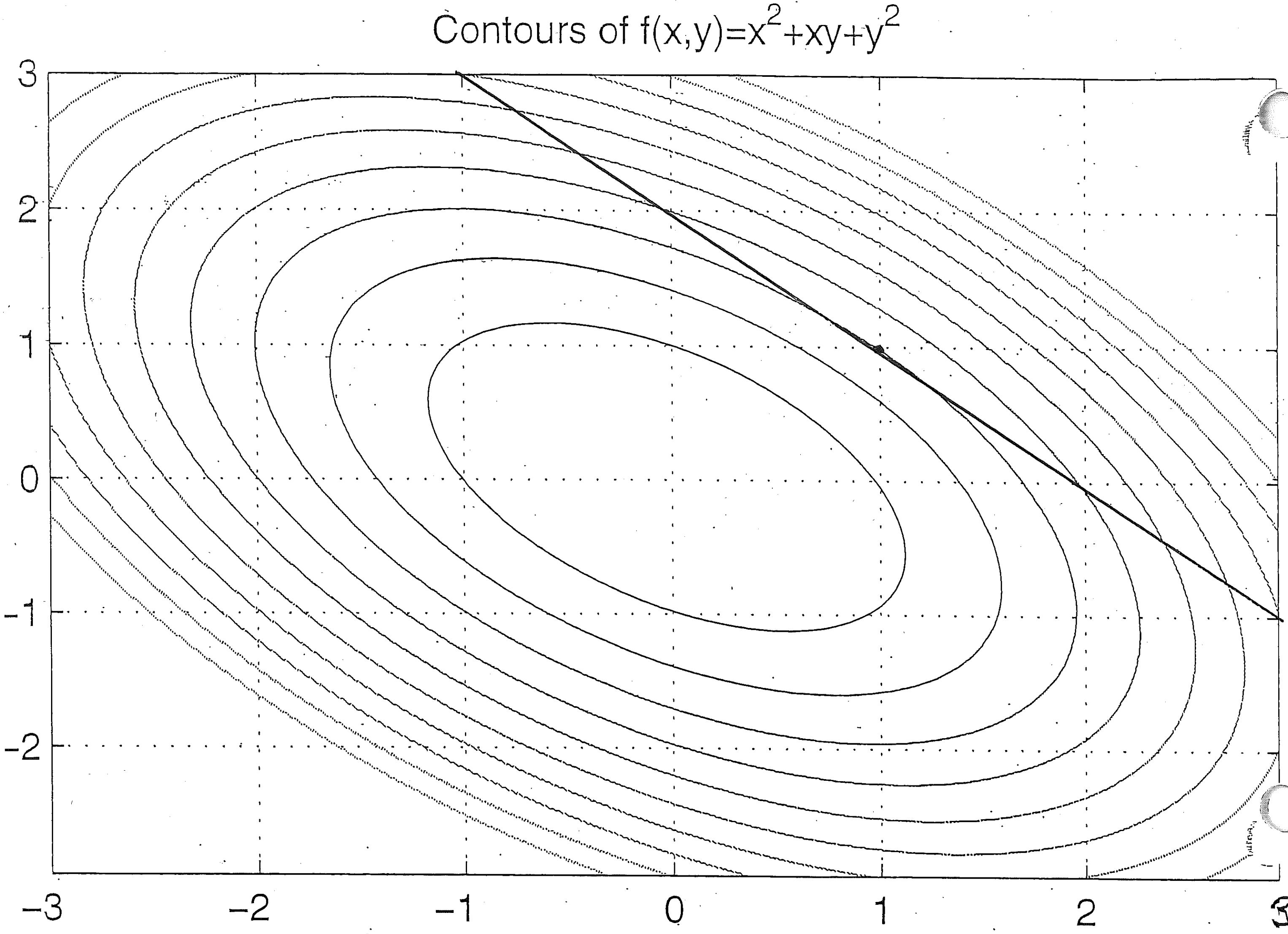
$f(\sqrt{2}, \sqrt{2}) = 2 + 2 + 2$
$= 6$

$f(-\sqrt{2}, \sqrt{2}) = 2 - 2 + 2 = 2$
$f(\sqrt{2}, -\sqrt{2}) = 2 - 2 + 2 = 2$
$f(-\sqrt{2}, -\sqrt{2}) = 2 + 2 + 2 = 6$

There is a global max at $(\sqrt{2}, \sqrt{2})$ & $(-\sqrt{2}, -\sqrt{2})$

There is a global min at $(-\sqrt{2}, \sqrt{2}, 2)$ & $(\sqrt{2}, -\sqrt{2}, 2)$

SPRING 2006



(e) Using the picture alone, estimate the points at which the objective function $f(x, y)$ achieves a global minimum on the constraint set $g(x, y) = 0$ and the values of f there. EXPLAIN YOUR ANSWER.

$g(x,y) = x + y - 2 = 0 \Rightarrow x + y = 2$

Looks like line intercepts the contour that goes through $x = 1, y = 1$

$f(1,1) = 1^2 + 1 \cdot 1 + 1^2 = 3$

The global min on the contour is 3.

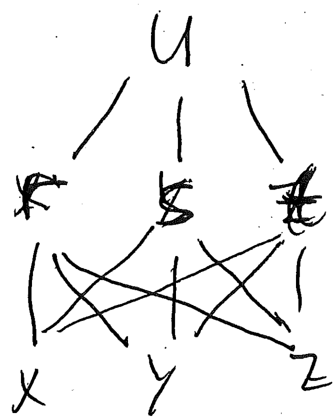
SPRING 2006

2. Multivariable Chain Rule. *25 points.*

Consider the functions $u(x, y, z) = f(x-y, y-z, z-x)$. Our goal is to show that a function u with this form satisfies the following famous partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

(a) (*10 points.*) Consider a function $u = f(r, s, t)$ where $r = r(x, y, z)$, $s = s(x, y, z)$ and $t = t(x, y, z)$ are given. In other words, although u is a function of r, s and t, since each of these functions is a function of x, y and z one can consider u as a function of x, y and z. Use the Chain Rule to write down expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$. [HINT: draw a "tree diagram" reflecting the relationships between the variables to assist you.]



$u_x = u_r r_x + u_s s_x + u_t t_x$

$u_y = u_r r_y + u_s s_y + u_t t_y$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

(b) (*15 points.*) Let $r = x-y$, $s = y-z$ and $t = z-x$. Use this information and your answer to (a) to show that $u(x, y, z) = f(x-y, y-z, z-x)$ satsfies the equation $u_x + u_y + u_z = 0$.

$r_x = 1$ $\quad s_x = 0$ $\quad t_x = -1$

$r_y = -1$ $\quad s_y = 1$ $\quad t_y = 0$

$r_z = 0$ $\quad s_z = -1$ $\quad t_z = 1$

$u_x = u_r \cdot 1 + u_s \cdot 0 + u_t \cdot -1 = u_r - u_t$

$u_y = u_r \cdot -1 + u_s \cdot 1 + u_t \cdot 0 = -u_r + u_s$

$u_z = u_r \cdot 0 + u_s \cdot -1 + u_t \cdot 1 = -u_s + u_t$

$u_x + u_y + u_z = \cancel{u_r} - \cancel{u_t} - \cancel{u_r} + \cancel{u_s} - \cancel{u_s} + \cancel{u_t}$

$= 0$

FALL 2005

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
Recall the Cobb-Douglas function $P(L,K) = bL^{\alpha}K^{1-\alpha}$ where the total production P of a certain product depends on the amount of labor L used and the amount K of capital investment ($0 < \alpha < 1$ and $b > 0$.)
If the cost of a unit of labor is m and the cost of unit of capital is n, given that the production of the company is fixed at a level Q, what values of L and K will minimize the cost function $C(L,K) = mL + nK$?

a. (*10 points*) Write down the equations you need to solve simultaneously to find the answer to the question.

Lagrange Multipliers

objective: $C(L,K) = mL + nK$

constraint: $Q = bL^{\alpha}K^{1-\alpha} \Rightarrow g(L,K) = bL^{\alpha}K^{1-\alpha} - Q$

$\vec{\nabla} C = \lambda \vec{\nabla} g$

$C_L = \lambda g_L \Rightarrow m = \lambda(\alpha b L^{\alpha-1} K^{1-\alpha})$

$C_K = \lambda g_K \qquad n = \lambda(bL^{\alpha}(1-\alpha)K^{-\alpha})$

$Q = bL^{\alpha}K^{1-\alpha}$

b. (*10 points*) Solve the equations to find the values of L and K which minimize the cost function $C(L,K)$. (**HINT: Eliminate the Lagrange Multiplier first**).

$$m = \lambda \alpha b \left(\frac{L}{K}\right)^{\alpha-1} \Rightarrow \frac{m}{\alpha b}\left(\frac{L}{K}\right)^{1-\alpha} = \lambda = \frac{n}{b(1-\alpha)}\left(\frac{L}{K}\right)^{-\alpha}$$

$$n = \lambda \cancel{b}(1-\alpha)\left(\frac{L}{K}\right)^{\alpha}$$

$$\frac{m}{\alpha}\frac{L}{K} = \frac{n}{\cancel{b}(1-\alpha)}$$

$$mL = Kn\frac{\alpha}{1-\alpha}$$

$$\frac{L}{K} = \frac{n\alpha}{m(1-\alpha)}$$

$$Q = b\left(\frac{L}{K}\right)^{\alpha}K$$

$$Q = b\left(\frac{n\alpha}{m(1-\alpha)}\right)^{\alpha}K$$

$$\boxed{\frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{-\alpha} = K} \Rightarrow L = \frac{n\alpha}{m(1-\alpha)}\cdot K = \frac{n\alpha}{m(1-\alpha)}\frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{-\alpha}$$

$$\boxed{L = \frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{1-\alpha}}$$