PRACTICE EXAM 2014

MATH 212

1. (a) $\vec{u} = 8\hat{\imath} - \hat{\jmath} + 4\hat{k}$ $\quad \|\vec{u}\| = \sqrt{8^2+1^2+4^2} = \sqrt{81} = 9$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{8}{9}\hat{\imath} - \frac{1}{9}\hat{\jmath} + \frac{4}{9}\hat{k}$$

(b) $P = (-1,3,2)$ $\quad \vec{a} = \begin{pmatrix}-1\\3\\2\end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix}2\\0\\4\end{pmatrix}$

$Q = (2,0,4)$

$$\vec{x} = \begin{pmatrix}-1\\3\\2\end{pmatrix} + t\left(\begin{pmatrix}2\\0\\4\end{pmatrix} - \begin{pmatrix}-1\\3\\2\end{pmatrix}\right) = \begin{pmatrix}-1\\3\\2\end{pmatrix} + t\begin{pmatrix}3\\-3\\2\end{pmatrix}$$

(c) $R = (1,1,1)$ is another point on plane

$$\vec{PR} = \begin{pmatrix}-1\\3\\2\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$\vec{QR} = \begin{pmatrix}2\\0\\4\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$$

Find normal to plane

$$\vec{n} = \vec{PR} \times \vec{QR} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -2 & 2 & 1\\ 1 & -1 & 3\end{vmatrix} = \hat{\imath}\,7 - \hat{\jmath}(-7) + 0\hat{k}$$

$$= 7\hat{\imath} + 7\hat{\jmath} + 0\hat{k}$$

$$\hat{n}\cdot x = \tilde{n}\cdot\vec{p}$$

$$\boxed{7x + 7y + 0z = 14}$$

2. (a)

$x = 5 - 12t$, $y = 3 + 9t$, $z = 1 - 3t$ or $x = 3 + 8s$, $y = -6s$, $z = 7 + 2s$

$\vec{d}_1 = \begin{pmatrix} -12 \\ 9 \\ -3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$ $\quad \vec{d}_2 = \begin{pmatrix} 8 \\ -6 \\ 2 \end{pmatrix} = -2\begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$

The lines are parallel because $\vec{d}_1 = k\vec{d}_2$, $k \in \mathbb{R}$

(b)

$$L = \int_{-5}^{5} \left\| \frac{d\vec{r}}{dt} \right\| dt \quad \text{where } \vec{r}(t) = \begin{pmatrix} t \\ 3\cos t \\ 3\sin t \end{pmatrix}$$

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} 1 \\ -3\sin t \\ 3\cos t \end{pmatrix}$$

$$L = \int_{-5}^{5} \sqrt{1^2 + (3\sin t)^2 + (3\cos t)^2}\, dt$$

$$= \int_{-5}^{5} \sqrt{19}\, dt = 10\sqrt{19}$$

3. (a) $F(x,y) = \int_y^x \cos(e^t)\, dt$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} \int_y^x \cos(e^t)\, dt = \cos(e^x)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} - \int_x^y \cos(e^t)\, dt = -\cos(e^y)$$

(b) $z = x^2 y^3$, $x = s\cos t$, $y = s\sin t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3 \cdot \cos t + 3x^2y^2 \cdot \sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2xy^3 \cdot (-s\sin t) + 3x^2y^2 \cdot (s\cos t)$$

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3(c) $yz + x\ln y = z^2$

$F(x,y,z) = yz + x\ln y - z^2 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-\ln y}{y-2z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-\left(z+\frac{x}{y}\right)}{y-2z}$$

4.(a) $z = 3y^2 - 2x^2 + x$ at $(2,-1,-3)$

$z = \cancel{-3} + f_x(2,-1)(x-2) + f_y(2,-1)(y+1)$

$z = f(x,y)$

$\frac{\partial f}{\partial x} = -4x + 1 \quad \frac{\partial f}{\partial y} = 6y$

At $x=2, y=-1, z=-3$

$f_x = -7 \quad f_y = -6$

$z = -3 + -7(x-2) - 6(y+1)$

$z = -7x + 14 - 3 - 6y - 6$

$7x + 6y + z = 5$

(b) $xy + yz + zx = 5$ at the point $(1,2,1)$

$f(x,y,z) = K \quad \vec{p} = (1,2,1)$

$\vec{\nabla} f = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$

$\vec{\nabla} f(1,2,1) = 3\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{\nabla} f(1,2,1)\cdot(\vec{x} - \vec{p}) = 0$

$$\begin{pmatrix}3\\2\\3\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-2\\z-1\end{pmatrix} = 0 \Rightarrow 3(x-1) + 2(y-2) + 3(z-1) = 0$$

$3x + 2y + 3z = 3 + 4 + 3 = 10$

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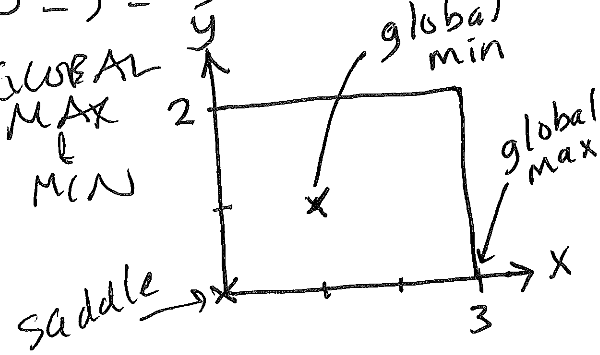
5. (a) $f(x,y) = x^4 + y^4 - 4xy + 2$

$D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$

D is closed & bounded so f MUST have GLOBAL MAX & MIN



$\vec{\nabla} f = \vec{0}$

$f_x = 4x^3 - 4y = 0 \Rightarrow x^3 = y$

$f_y = 4y^3 - 4x = 0 \Rightarrow y^3 = x$

$f_{xx} = 12x^2$ $\qquad (x^3)^3 = x$

$f_{yy} = 12y^2$ $\qquad x^9 - x = 0 \Rightarrow x = 0$ or $x^8 = 1$, $x = \pm 1$

$f_{xy} = -4$ $\qquad (x^8 - 1)x = 0$

Sol^'s are $(0,0)$, $(1,1)$ and $(-1,-1)$

Since $(-1,-1)$ is not in D we ignore it

SADDLE $f(0,0) = 2$ $\qquad f(+1,+1) = 1 + 1 - 4(+1) + 2 = 0$ LOCAL MIN

$D(1,1) = 128 > 0$

$x = 0$ border

$h(y)$ $\quad f(0,y) = y^4 + 2$ for $0 \le y \le 2$

$h'(y) = 4y^3 = 0 \Rightarrow y = 0$ $\qquad h(0) = 2$ is local min

$h(2) = 18$ is local max

$x = 3$ border $\quad f(3,y) = 3^4 + y^4 - 12y + 2$ for $0 \le y \le 2$

$g(y) = y^4 - 12y + 83$ for $0 \le y \le 2$

$g'(y) = 4y^3 - 12 = 0 \Rightarrow y^3 = 3 \Rightarrow y = \sqrt[3]{3}$

$g''(y) = 12y^2$ $\qquad g''(3^{1/3}) = 12\, 3^{2/3} > 0 \Rightarrow$ min

$g(3^{1/3}) = 3^{1/3}(-9) + 83$ $\qquad g(0) = 83$ local max $\qquad g(2) = 99 - 24 = 75$

$y = 0$ border

$f(x,0) = x^4 + 2$, $0 \le x \le 3$ $\qquad$ min at $x = 0$, max at $x = 3$ $\qquad f(0,0) = 2$, $f(3,0) = 83$ local max & global max

$y = 2$ border

$f(x,2) = x^4 + 2^4 - 4x \cdot 2 + 2$

$= x^4 + 16 - 8x + 2$

$g(x) = x^4 + 18 - 8x$, $0 \le x \le 3$ $\qquad x^3 = 2$

$g'(x) = 4x^3 - 8 = 0 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2}$ local

$g''(x) = 12x^2$ $\qquad g''(2^{1/3}) > 0$ so this is a min

$g(0) = 18$ $\qquad g(3) = 75$

$g(2^{1/3}) = x(x^3 - 8) + 18$

$= 2^{1/3}(-6) + 18$

YOUR PROBLEM WOULD NOT BE THIS HARD ☺

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5(b) $z^2 = x^2 + y^2 \Rightarrow g(x,y,z) = x^2+y^2-z^2 = 0$

$D^2 = (x-4)^2 + (y-2)^2 + (z-0)^2 = f(x,y,z)$

$\vec{\nabla} f = \lambda \vec{\nabla} g$
$g = 0$

$f_x = \lambda g_x$
$f_y = \lambda g_y$
$f_z = \lambda g_z$
$x^2+y^2 = z^2$

$2(x-4) = \lambda 2x$
$2(y-2) = \lambda 2y$
$2(z-0) = -\lambda 2z$
$\lambda = -1$
So $2z = 2z$

$2x - 8 = -2x$
$4x = 8 \Rightarrow x = 2$
$2y - 4 = -2y$
$4y = 4 \Rightarrow y = 1$
$z^2 = 1^2 + 2^2 = 5 \Rightarrow z = \pm\sqrt{5}$

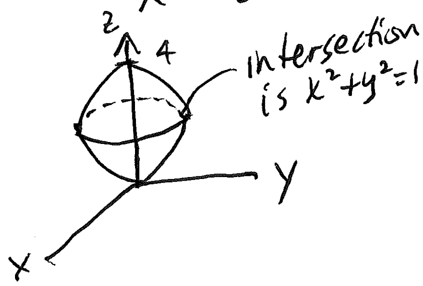
points are $(2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$ closest to $(4,2,0)$.

6(a) $\displaystyle\int_{-3}^{3}\int_{0}^{1} \frac{xy^2}{x^2+1}\,dx\,dy = \int_{-3}^{3} \frac{1}{2}\ln(x^2+1)\Big|_0^1 y^2\,dy = \int_{-3}^{3}\left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right)y^2\,dy$

$\displaystyle = \int_{-3}^{3} \frac{1}{2}\ln 2\, y^2\,dy = \frac{1}{2}\ln 2\left(\frac{y^3}{3}\right)\Big|_{-3}^{3}$

$\displaystyle = \frac{1}{2}\ln 2\,(9 - -9) = 9\ln 2$

6(b) $z = 3x^2+3y^2$
$z = 4 - x^2 - y^2$
$3x^2+3y^2 = 4 - x^2 - y^2$
$4x^2+4y^2 = 4$
$x^2+y^2 = 1$



Volume

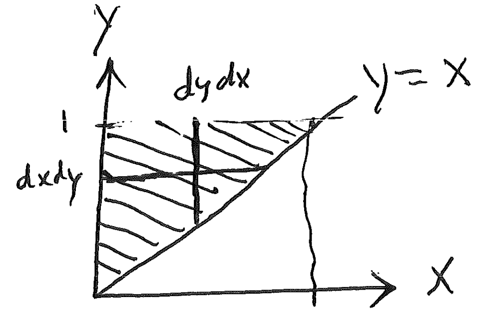
$\displaystyle\int_0^{2\pi}\int_0^1\int_{3r^2}^{4-r^2} dz\,r\,dr\,d\theta$

$\displaystyle = 2\pi\int_0^1 (4 - r^2 - 3r^2)\,r\,dr$

$\displaystyle = 2\pi\int_0^1 4(r - r^3)\,dr = 8\pi\int_0^1 r - r^3\,dr$

$\displaystyle = 8\pi \left.\frac{r^2}{2} - \frac{r^4}{4}\right|_0^1 = 8\pi\cdot\frac{1}{4} = 2\pi$

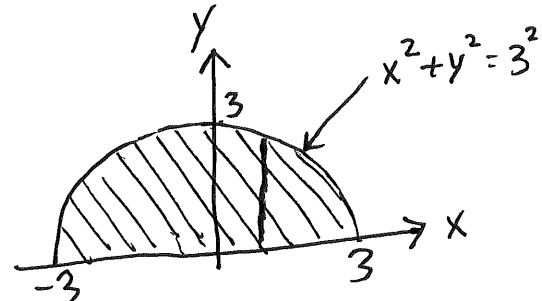
7(a) $\displaystyle\int_0^1 \int_x^1 e^{\frac{x}{y}}\,dy\,dx$



$\displaystyle = \int_0^1 \int_0^y e^{\frac{x}{y}}\,dx\,dy = \int_0^1 y e^{x/y}\Big|_{x=0}^{x=y} dy = \int_0^1 ye^1 - ye^0\,dy$

$\displaystyle = \int_0^1 (e-1)y\,dy = (e-1)\int_0^1 y\,dy = (e-1)\frac{1}{2}$

7(b) $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$



$\displaystyle = \int_0^3 \int_0^{\pi} \sin(r^2)\,r\,d\theta\,d\theta = \int_0^{\pi} \int_0^3 \sin(r^2)\,r\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi} -\frac{\cos(r^2)}{2}\Big|_0^3 d\theta = \left(-\frac{\cos(9)}{2} - \frac{-\cos 0}{2}\right)\cdot 2\pi$

$= \pi(1-\cos(9))$

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8. (a) $\vec{F}(x,y,z) = (x - y^2)\hat{i} + (y - z^2)\hat{j} + (z - x^2)\hat{k}$

$\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

C: Eq^n of line from $\vec{a}$ to $\vec{b}$: $\vec{x} = \vec{a} + (\vec{b} - \vec{a})t, \ 0 \le t \le 1$

$\vec{x}(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} t$

$\vec{x}(t) = \begin{pmatrix} 2t \\ t \\ 1 - t \end{pmatrix} \Rightarrow \dfrac{d\vec{x}}{dt} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$$\int_C \vec{F} \cdot d\vec{x} = \int_0^1 \vec{F}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}\, dt$$

$$= \int_0^1 \begin{pmatrix} x - y^2 \\ y - z^2 \\ z - x^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} 2t - t^2 \\ t - (1-t)^2 \\ 1 - t - (2t)^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} dt$$

$$= \int_0^1 2 \cdot (2t - t^2) + 1 \cdot (t - (1-t)^2) - 1 \cdot (1 - t - 4t^2)\, dt$$

$$= \int_0^1 4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2\, dt$$

$$= \int_0^1 -2 + 8t + t^2\, dt = -2t + 4t^2 + \frac{t^3}{3} \Big|_0^1$$

$$= -2 + 4 + \frac{1}{3} - 0 = \boxed{\frac{7}{3}}$$

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8b Find f so that $\vec{F} = \vec{\nabla} f$
where $\vec{F} = xy^2\hat{\imath} + x^2y\hat{\jmath}$
and use this to compute $\int_C \vec{F}\cdot d\vec{r}$
where $C: \vec{r}(t) = \begin{pmatrix} t + \sin\frac{\pi}{2}t \\ t + \cos\frac{\pi}{2}t \end{pmatrix}$, $0 \le t \le 1$

This is a Gradient Field so use FTLI.
First got to find f.

$$\vec{\nabla} f = f_x\hat{\imath} + f_y\hat{\jmath} = xy^2\hat{\imath} + x^2y\hat{\jmath}$$

$$f_x = xy^2 \Rightarrow f = \tfrac{1}{2}x^2y^2 + C(y)$$

$$f_y = x^2y \qquad f_y = x^2y + C'(y) = x^2y$$

so $C'(y) = 0$

$$f = \tfrac{1}{2}x^2y^2$$

$$\int \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(\vec{b}) - f(\vec{a})$$

$$\vec{a} = \vec{r}(0) = \begin{pmatrix} 0 + \sin\frac{\pi}{2}0 \\ 0 + \cos\frac{\pi}{2}0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{b} = \vec{r}(1) = \begin{pmatrix} 1 + \sin\frac{\pi}{2}1 \\ 1 + \cos\frac{\pi}{2}1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\int \vec{F}\cdot d\vec{r} = f\begin{pmatrix} 2 \\ 1 \end{pmatrix} - f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\cdot 2^2\cdot 1^2 - \frac{1}{2}0^2\cdot 1^2 = \boxed{2}$$

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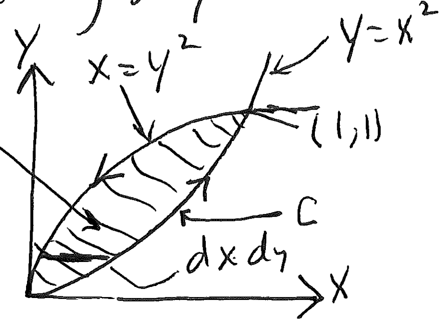
9. Use Green's Theorem to evaluate

$$\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

where C is boundary of R



$$\vec{F} = \begin{pmatrix} y+e^{\sqrt{x}} \\ 2x+\cos y^2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Green's Theorem

$$\int_C \vec{F}\cdot d\vec{x} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(2x+\cos y^2) = 2 \qquad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(y+e^{\sqrt{x}}) = 1$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (2-1)\,dA = \iint_R dA$$

This is an area problem

$$\text{Area} = \int_0^1\int_{y^2}^{\sqrt{y}} dx\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx$$

$$= \int_0^1 \sqrt{y} - y^2\,dy = \left(\frac{2}{3}y^{3/2} - \frac{y^3}{3}\right)\Bigg|_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0)$$

$$= \boxed{\frac{1}{3}}$$

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10. $\vec{F} = y\hat{j} - z\hat{k}$

Find $\text{curl}\,\vec{F}$ and $\text{div}\,\vec{F}$

Recall if $\vec{F} = F_1\hat{\imath} + F_2\hat{j} + F_3\hat{k}$

$\text{div}\,\vec{F}$ $\quad \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

so $\vec{\nabla}\cdot\begin{pmatrix}0\\ y\\ -z\end{pmatrix} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-z)$

$= 0 + 1 - 1$

$\text{div}\,\vec{F} = 0$

$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix}\hat{\imath} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$

$= \begin{vmatrix}\hat{\imath} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & y & -z\end{vmatrix}$

$= \hat{\imath}\left(\frac{\partial}{\partial y}(-z) - \frac{\partial}{\partial z}(y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial z}(0)\right)$

$+ \hat{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(0)\right)$

$= \hat{\imath}\cdot 0 - \hat{j}\cdot 0 + \hat{k}\cdot 0$

$\text{curl}\,\vec{F} = \vec{0}$