# Multivariable Calculus


 

*Class 31*: **Monday May 1**

**SUMMARY** Conservative Vector Fields
**CURRENT READING** Williamson & Trotter, §9.2
**HOMEWORK** Williamson & Trotter, page 418: **19, 20, 21, 22, 23 Extra Credit page 419: 27**.

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THEOREM

All gradient fields are conservative vector fields. All conservative vector fields have zero curl. All gradient fileds have zero curl.

THEOREM: properties of conservative vector fields

Let $\vec{F}$ be a continuous vector field defined in a polygonally connected open set $D$ in $\mathbb{R}^n$. THEN *each* of the following three statements implies the other two.
**(a)** The integral $\vec{F}(\vec{x})$ over every piecewise smooth path from $\vec{x}_1$ to $\vec{x}_2$ in $D$ has the same value, and we can write it as $\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = \int_{\vec{x}_1}^{\vec{x}_2} \vec{\nabla} f(\vec{x}) \cdot d\vec{x} = f(\vec{x}_2) - f(\vec{x}_1)$.
**(b)** The integral over every piecewise smooth closed path $\gamma$ in $D$ is **zero**. In other words $\oint_\gamma \vec{F} \cdot d\vec{x} = \oint_\gamma \vec{\nabla} f \cdot d\vec{x} = 0$
**(c)** There is a continously differentiable function $f : D \to \mathbb{R}$ such that $\vec{F}$ is the gradient of $f$, i.e. $\vec{\nabla} f = \vec{F}$ for all $\vec{x}$ in $D$.

THEOREM

IF $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable gradient field, then $\vec{F}_{\vec{x}}$, the Jacobian matrix of $\vec{F}$ is **symmetric**. In other words $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all $i, j = 1, 2, \ldots, n$.

EXAMPLE 1

**Williamson & Trotter, page 418, #3.** Is $\vec{F}(x, y) = (x - y, x + y)$ a gradient field?

**Exercise 2**

**Williamson & Trotter, page 418, #4.** Is $\vec{G}(x, y, z) = (y, z, x)$ a gradient field?

EXAMPLE 2

**Williamson & Trotter, page 418, #10.** Find a field potential for the given field.
$\vec{F}(x, y) = (2xy, x^2 + z^2, 2yz)$.

**Exercise 2**

**Williamson & Trotter, page 418, #11.** Find a field potential for the given field.
$\vec{G}(x, y) = (y\cos(xy), x\cos(xy))$.

GroupWork

**Williamson & Trotter, page 418, #14.** Consider the vector field $\vec{F}$ which is the gradient of the **Newtonian potential** $f(\vec{x}) = -|\vec{x}|^{-1}$ for nonzero $\vec{x}$ in $\mathbb{R}^3$. Find the work done in moving a particle from $(1, 1, 1)$ to $(-2, -2, -2)$ along a smooth curve lying in the domain of $\vec{F}$.