Multivariable Calculus

Math 212 Spring 2006 Fowler 112 MWF 8:30pm - 9:25am

http://faculty.oxy.edu/ron/math/212/06/

Class 28: **Wednesday April 19**

SUMMARY Div, Grad, Curl and all that!
CURRENT READING Williamson & Trotter, §8.4
HOMEWORK page 394: 1, 2, **4, 6, 8 Extra Credit page 394-395: 11, 19, 20, 21**

DEFINITION: divergence

The **divergence** of a vector field $\vec{F}(\vec{x})$ is a scalar property denoted by $\mathbf{div}\vec{F}(\vec{x})$ defined as the trace of $\vec{F}_{\vec{x}}$ (the Jacobian matrix), i.e. the sum of the diagonal elements of this matrix. In particular, if one considers $\vec{F} = (F_1, F_2, F_3)$ in $\mathbb{R}^3$ where $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ then $\mathbf{div}\vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. The divergence is the **dot product** of the gradient operator with a vector field $\vec{F}$, so it is a scalar quantity.

DEFINITION: curl

The **curl** of a vector field $\vec{F}(\vec{x})$ is a vector property denoted by $\mathbf{curl}\vec{F}(\vec{x})$ and defined as $\vec{\nabla} \times \vec{F}$ where $\vec{F} = (F_1, F_2, F_3)$ in $\mathbb{R}^3$ and $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

$$\mathbf{curl}\vec{F} = \vec{\nabla} \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

The curl is the **cross product** of the gradient operator with a vector field $\vec{F}$, so it is a vector quantity.

EXAMPLE 1

Williamson & Trotter, page 394, #10. Consider $\vec{F}(x, y, z) = (x^3, -y^3, z^3)$. Compute **div** $\vec{F}$ and **curl** $\vec{F}$.

$\boxed{\textbf{Exercise 1}}$

Williamson & Trotter, page 394, #9. Consider $\vec{F}(x,y) = (e^{x+y}, e^{x-y})$. Compute **div** $\vec{F}$ and **curl** $\vec{F}$, the divergence and curl of this vector field. Why can't you compute **grad** $\vec{F}$? Is there a problem computing **curl** $\vec{F}$?

$\boxed{\text{DEFINITION: scalar curl}}$

Given a vector field $\vec{F}(x,y,z) = (F_1(x,y), F_2(x,y), 0)$ by definition the **curl** of $\vec{F}$ is given by $(0, 0, \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y})$. Similarly, given a 2-D vector field $\vec{F}(x,y) = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ which we usually write as $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$ we can define the (badly-misnamed) **scalar curl** of $\vec{F}$ to be

$$\textbf{curl } \vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Our Favorite Vector Fields

Recall the vector fields $\vec{F}(x,y) = (x,y)$ and $\vec{G}(x,y) = (-y,x)$. What do these vector fields look like? How can we match their appearance with their (scalar) **curl** and **div**?

$\boxed{\textbf{Exercise 2}}$

Sketch the vector fields $\vec{F}$ and $\vec{G}$ and find their divergence and curl.

GroupWork Activity

To help us visualize vector fields we can use the **Paul Falstad's Vector Field Simulator** at `http:/falstad.com/vector`. These links are also available on the Math 212 Resources webpage.

Consider $\vec{H}(x, y; s) = ((1-s)x - sy, sx + (1-s)y)$. Notice that when $s = 0$, $\vec{H} = \vec{F}$ and when $s = 1$, $\vec{H} = \vec{G}$. So s is a parameter which moves one from vector field F to vector field G.

Question 1 What do you expect to happen to the **curl** and **div** of $\vec{H}$ as s varies from 0 to 1? (Sketch what you see!)

Question 2 Use the Vector Field Simulator to look at 3 different vector fields: one with (i) **zero curl** and **non-zero divergence**; (ii) **non-zero curl** and **zero divergence**; and (iii) **non-zero curl** and **non-zero divergence**. (Sketch what you see!)

Question 3 Use the Vector Field Simulator to compute a **Line Integral** and an **Area Integral** for each of the Vector Fields in Question 2. What relationship do you notice between the value of the integrals and the existence of the curl and divergence of each vector field?

(Introduction to Fluid Dynamics) Applications of Divergence and Curl

Divergence and Curl are two properties of a vector field. Generally, Divergence relates to the expansion/contraction rate of the fluid at a point in space per unit area/volume. In fluid dynamics, one of the basic equations is known as the **continuity equation** which is often expressed as

$$\vec{\nabla} \cdot (\rho \vec{v}) = -\frac{\partial \rho}{\partial t}$$

where $\rho(\vec{x}, t)$ is the density of the fluid at any point in space-time and $\vec{u}(\vec{x}, t)$ is the velocity vector at any point in space-time. For an **incompressible** fluid (like water, and air at most standard aircraft operating temperatures) the continuity equation becomes simply that $\vec{\nabla} \cdot \vec{v} = 0$.

The curl of a vector field is a measure of the tendency for the flow lines to rotate about an axis. Again, in fluid dynamics, we often deal with flows that are **irrotational**, in other words $\vec{\nabla} \times \vec{v} = \vec{0}$. This is because in this case we get the special case of **potential flow** where $\vec{v}$ can be represented as $\vec{\nabla}\phi(\vec{x}, t)$, since by definition $\vec{\nabla} \times \vec{\nabla}\phi = 0$. In other words, **the curl of a gradient field is by definition zero**.

There's a whole field of mathematics and engineering which involves looking at incompressible, irrotational vector fields, called hydrodynamics, and some aspects of aerodynamics are also included in this area of potential theory.

EXAMPLE 2

Let's prove that $\vec{\nabla} \times \vec{\nabla}\phi = 0$

Exercise 3

Show that an incompressible, irrotational fluid must possess a velocity potential which satisfies Laplace's Equation (i.e. $\Delta\phi = \nabla^2\phi = 0$). [HINT: if a flow is irrotational what does that tell you about **curl** $\vec{v}$ where $\vec{v}(\vec{x})$ is the fluid velocity at any point in space $\vec{x}$ in $\mathbb{R}^3$, and what does *that information* tell you about what kind of field $\vec{v}(\vec{x})$ is?]