Multivariable Calculus

Math 212 Spring 2006 | Fowler 112 MWF 8:30pm - 9:25am

 | http://faculty.oxy.edu/ron/math/212/06/

Class 25: **Wednesday April 12**

SUMMARY Change of Variables Theorem
CURRENT READING Williamson & Trotter, Section 7.4
HOMEWORK #24 5,11,16,17,21,**35,40,43,44**,49 **Extra Credit page page 365: 48, 50**

Suppose we want to change variables from an integral defined over $T(R)$ over one that is defined over R where $\vec{x} = \vec{T}(\vec{u}) : \mathbb{R}^n \to \mathbb{R}^n$

Jacobi's Theorem

Given $\vec{T} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable transformation and R is a subset of $\mathbb{R}^n$ having a boundary consisting of finitely many smooth sets. IF R and its boundary are contained in the domain of $\vec{T}$ and that **(i)** $\vec{T}$ is one-to-one on the interior of R and **(ii)** det $(\vec{T}_{\vec{u}}) \neq 0$ in the interior of R, THEN

$$\int_{T(R)} f(\vec{x})\, dV_x = \int_R f(\vec{T}(\vec{u}))\, |\det(\vec{T}_{\vec{u}}(\vec{u}))|\, dV_u$$

or, using Leibnizian Notation where T maps from W^* in uvw-space to $W = T(W^*)$ in xyz-space

$$\int\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw \text{ in } \mathbb{R}^3$$

and

$$\int\int_W f(x,y)\, dx\, dy = \int\int_{W^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \text{ in } \mathbb{R}^2$$

Generally, we use this theorem to convert from Cartesian coordinates to polar, spherical, and cylindrical co-ordinates.

Change of Variables: Polar Coordinates

$$\int\int_D f(x,y)\, dx\, dy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Change of Variables: Cylindrical Coordinates

$$\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Change of Variables: Spherical Coordinates

$$\int\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

EXAMPLE 1

Williamson & Trotter, page 346, #7. Compute $\int_D \cos(x^2 + y^2)\ dx\ dy$ where D is the disk of radius $\sqrt{\pi/2}$ centered at (0,0).

Exercise 1

Williamson & Trotter, page 346, #12. Compute $\int_C z^2\ dx\ dy\ dz$ where C is the region in $\mathbb{R}^3$ described by $1 \leq x^2 + y^2 + z^2 \leq 4$

Paired GroupWork

Williamson & Trotter, page 347, #21. Consider the transformation T defined by $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u^2 - v^2 \\ 2uv \end{bmatrix}$. Let R_{uv} be the region $1 \le u^2 + v^2 \le 4, \; u \ge 0, \; v \ge 0$.

(a). Sketch the image region $R_{xy} = T(R_{uv})$. **(b)** Compute $\displaystyle\int_{R_{xy}} \frac{dx\, dy}{\sqrt{1 + 4x + 4y}}$