Multivariable Calculus

Math 212 Spring 2006 — Fowler 112 MWF 8:30pm - 9:25am

 — http://faculty.oxy.edu/ron/math/212/06/

Class 19: **Wednesday March 22**

SUMMARY Implicit Differentiation
CURRENT READING Williamson & Trotter, Section 6.3
HOMEWORK Williamson & Trotter, page 274: 2, **3**; page 281: 2, 3, **4**, 5, **7**, 12, **15**

THEOREM: The Inverse Function Theorem or I.F.T.

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open subset S of $\mathbb{R}^n$ and let $\vec{x}_0$ be a point in S with invertible derivative matrix (Jacobian) $\vec{F}'$. THEN there is a neighborhood N of $\vec{x}_0$ such that $\vec{F}$ has a continuously differentiable inverse function $(\vec{F})^{-1}$ defined on the image set $\vec{F}(N)$. The derivative matrix (Jacobian) of $\vec{F}^{-1}$ is related to the Jacobian matrix of $\vec{F}$ by the equation

$$[\vec{F}^{-1}]'(\vec{F}(\vec{x})) = [\vec{F}'(\vec{x})]^{-1}$$

You should read this as the derivative of the inverse function of $\vec{F}$ evaluated at $\vec{F}$ is equal to the inverse of the derivative of the function $\vec{F}$ evaluated at $\vec{x}$.

This is the multivariable version of the common result from Calculus you may recall that given $f(a) = b$ and $a = g(b)$ so that g is the inverse function of f, then
$g'(f(a)) = g'(b) = \dfrac{1}{f'(a)}$

If we think of these objects as vectors we can let $\vec{g} = \vec{f}^{-1}$ and $\vec{a} = \vec{g}(\vec{b}) \leftrightarrow \vec{b} = \vec{f}(\vec{a})$ then we can re-write the I.F.T. as $\mathbf{g}'(\vec{b}) = [\mathbf{f}'(\vec{a})]^{-1}$ where $\mathbf{g} = \vec{g}$ and $\mathbf{f} = \vec{f}$.

Exercise 1

$f(x) = e^x$ and $g(x) = \ln(x)$ then f and g are inverse functions of each other. Let $a = 1$ and $b = f(a) = e^1 = e$. Compute $g'(e)$ using the Inverse Function Theorem (i.e. without differentiating $g(x)$)

The main usefulness of the Inverse Function Theorem is when one is doing a **coordinate transformation**. This will be extremely more important when we look at Multivaiable Integration later on.

EXAMPLE 1

Williamson & Trotter, page 274, #10. Define $P : \mathbb{R}^2 \to \mathbb{R}^2$ by the equations $x = u\cos v, y = u\sin v$ for $u > 0$.

(a) Show that for fixed $v = v_0$ and varying $u > 0$ the image curves in the xy-plane are half-lines emanating from $(x, y) = (0, 0)$. (QUESTION: What do the pre-image curves look like in the uv-plane?)

(b) Show that for fixed $u = u_0$ and varying v the image curves in the xy-plane are circles of radius u_0 each one traced infinitely often. (QUESTION: What do the pre-image curves look like in the uv-plane?)

(c) Compute the determinant of the jacobian matrix $P'(u, v)$, sometimes denoted $\dfrac{\partial(x, y)}{\partial(u, v)}$, and show that if $u_0 \neq 0$ then the inverse function theorem implies the existence of a local inverse in the neighborhood of $(x_0, y_0) = P(u_0, v_0)$.

Implicit Differentiation

Recall that even though one does not always have an explicit definition of a curve $y = f(x)$ one can still compute the slope of such a curve using a process called **implicit differentiation**. The curve is defined *implicitly* as $F(x, y) = c$.

Using the Chain Rule this equation becomes $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ when when solved implies that

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -(F_y)^{-1}F_x$$

EXAMPLE 2

Show that the implicitly-defined curve $y^3 - xy = -6$ has no points on it where a tangent line would have zero slope.

Exercise 2

The equation $F(x, y) = x^2 + y^2 - 4 = 0$ defines a circle of of radius 2 centered at the origin. What's the slope of this curve at $(1, \sqrt{3})$? What's the slope at $(2, 2)$?

Are there any points on this curve where the slope of this curve has a *horizontal* tangent line?

Are there any points on this curve where the slope of this curve has a *vertical* tangent line?

THEOREM: The Implicit Function Theorem

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function. Suppose for some $\vec{x}_0$ in $\mathbb{R}^n$ and some $\vec{y}_0$ in $\mathbb{R}^m$ that
(i) $\vec{F}(\vec{x}_0, \vec{y}_0) = \vec{0}$ and
(ii) $\vec{F}_{\vec{y}}(\vec{x}_0, \vec{y}_0)$ is an m-by-m invertible matrix.
THEN there is a unique continuously differentiable function $G : \mathbb{R}^n \to \mathbb{R}^m$ defined on an open neighborhood N of $\vec{x}_0$ in $\mathbb{R}^n$ such that $\vec{F}(\vec{x}, \vec{G}(\vec{x})) = \vec{0}$ for all $\vec{x}$ in N and $\vec{G}(\vec{x}_0) = \vec{y}_0$.

Theorem

Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable and that $\vec{y} = \vec{G}(\vec{x})$ satisfies $\vec{F}(\vec{x}, \vec{y}) = \vec{0}$ for all $\vec{x}$ in some open subset of $\mathbb{R}^n$. THEN

$$\vec{G}'(\vec{x}) = -[\vec{F}_{\vec{y}}]^{-1}(\vec{x}, \vec{G}(\vec{x}))\vec{F}_{\vec{x}}(\vec{x}, \vec{G}(\vec{x}))$$

This is the multi-dimensional version of finding the slope of an implicitly-defined curve $y(x)$ which satisfies $f(x, y) = c$, i.e. implicit differentiation. In multi-dimensions we're trying to find an $m \times n$ derivative matrix corresponding to the "rate of change" of an implicitly defined m-component vector function of an n-component input variable. Do you see how this compares to the expression for $\dfrac{dy}{dx}$ of an implicitly defined curve $F(x, y) = c$ on Page 2?

Exercise 3

Williamson & Trotter, page 281, #8. If $x + y - u - v = 0$ and $x - y + 2u + v = 0$. Find $\partial x/\partial u$,$\partial y/\partial u$, $\partial x/\partial v$ and $\partial y/\partial v$ by (1) solving for x and y in terms of u and v and (2) by implicit differentiation.

EXAMPLE 3

Consider **Williamson & Trotter, page 281, #7**. Suppose $x^2y + yz = 0$ and $xyz + 1 = 0$. **(a)** Find dx/dz and dy/dz at $(x, y, z) = (1, 1, -1)$. **(b)** Find dy/dx and dz/dx at $(x, y, z) = (1, 1, -1)$. **(c)** Find dx/dy and dz/dy at $(x, y, z) = (1, 1, -1)$.