Test 3: Multivariable Calculus

Math 212 | Friday April 28 2006
Prof. Ron Buckmire | 8:30pm-9:25am

Name: Key

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. Notice which skills are designed to be tested on each question. This is a one hour, limited-notes (1 page), closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ____________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		30
2		40
3		30
BONUS		10
Total		**100**

1. Vector Fields, Double Integrals, Triple Integrals. *30 points.*
SKILLS: ANALYSIS/CRITICAL THINKING, VERBAL EXPRESSION.

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once. .

(a) *10 points.* **TRUE or FALSE?** "There exists a non-zero vector field in $\mathbb{R}^3$ which has both zero curl and zero divergence."

$\boxed{\text{TRUE}}$

$$\vec{F}(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{\nabla} \times \vec{F} = \vec{0}$$
$$\vec{\nabla} \cdot \vec{F} = 0$$

Since $\vec{F}(\vec{x})$ is constant, all derivatives of the components will equal zero.

(b) *10 points.* **TRUE or FALSE?** "Every triple integral can be written as a double integral."

$\boxed{\text{FALSE}}$

$\int_0^1 \int_0^y \int_0^2$ ~~f(x,y,z) dzdxdy can not~~ be wri~~tten as a double integ~~ral ~~because~~ ~~there's no way to represent the f(x,y,z) in~~ $\cancel{\mathbb{R}^2}$

Only true if Fubini is always true, i.e. $\iiint f(x,y,z)\,dx\,dy\,dz$ is unintegrable, so have to switch $\iiint f\,dz\,dy\,dx$. May not ALWAYS be possible.

(c) *10 points.* **TRUE or FALSE?** "Every double integral can be written as a triple integral."

$\boxed{\text{TRUE}}$

$$\int_a^b \int_{u(y)}^{v(y)} f(x,y)\,dx\,dy = \int_a^b \int_{u(y)}^{v(y)} \int_0^{f(x,y)} dz\,dx\,dy$$

2. Line Integration, Iterated Integration, Multiple Integration. (*40 points.*)
SKILLS: VISUALIZATION, COMPUTATION.

(a) (*10 points.*) Consider the path γ to be the straight line from the point (x_1, y_1) to the point (x_2, y_2). Write down a parametrization of this path and then show that the value of the line integral $\int_\gamma -\frac{y}{2}\, dx + \frac{x}{2}\, dy = \frac{1}{2}(x_1 y_2 - x_2 y_1)$. **[HINT: be very meticulous about maintaining all the subscripts during this calculation!]**

$$\vec{x}(t) = \vec{a}(1-t) + \vec{b}t, \quad 0 \le t \le 1$$

$$\vec{x}(0) = a, \quad \vec{x}(1) = \vec{b} \qquad \vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\frac{d\vec{x}}{dt} = \vec{b} - \vec{a}$$

$$\vec{F}(\vec{x}) = \begin{pmatrix} -y/2 \\ x/2 \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(1-t) + x_2 t \\ y_1(1-t) + y_2 t \end{pmatrix}$$

$$I = \int_\gamma \vec{F} \cdot d\vec{x} = \frac{1}{2}\int_0^1 \begin{pmatrix} -y_1(1-t) - y_2 t \\ x_1(1-t) + x_2 t \end{pmatrix} \cdot \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} d$$

THESE CANCEL

$$I = \frac{1}{2}\int_0^1 (x_2 - x_1)(y_1 - y_2)t + (y_2 - y_1)(x_2 - x_1)t - y_1(x_2 - x_1) + (y_2 - y_1)x_1 \, dt$$

$$= \frac{1}{2}\int_0^1 -y_1(x_2 - x_1) + (y_2 - y_1)x_1 \, dt$$

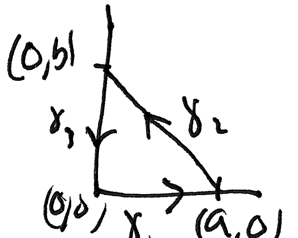
$$= \frac{1}{2}\left[-y_1 x_2 + y_1 x_1 + y_2 x_1 - y_1 x_1\right]$$

$$= \frac{1}{2}\left[x_1 y_2 - y_1 x_2\right]$$

Γ

(b) (*10 points.*) Consider a triangular path Γ formed in the first quadrant by first moving from the origin horizontally a units to the right along the x-axis and then from this point moving upwards diagonally to the point b units on the y-axis above the origin and then down vertically back to the origin. Draw a picture of this path, labeling the coordinates of the vertices. Use your result from part (a) to help you in evaluating $\int_\Gamma -\frac{y}{2}\, dx + \frac{x}{2}\, dy$. **[HINT: you should not have to actually DO any integration in this problem!]**

$$\int_\Gamma -\frac{y}{2}\, dx + \frac{x}{2}\, dy = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$



$$= \frac{1}{2}[0 \cdot 0 - a \cdot 0] + \frac{1}{2}[a \cdot b - 0 \cdot 0] + \frac{1}{2}[0 \cdot 0 - b \cdot 0]$$

$$= \frac{1}{2} 0 + \frac{1}{2} ab + \frac{1}{2} 0$$

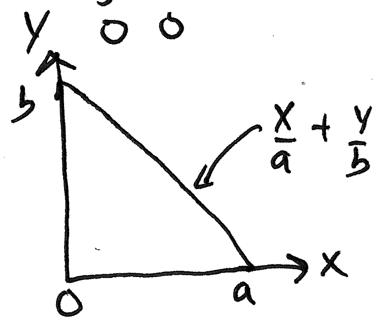
$= \frac{1}{2}ab$ = area of Δ enclosed by Γ

(c) (*10 points.*) Considering Green's Theorem, write down a double integral which is equal in value to the line integral computed in part (b). Reverse the order of integration and evaluate this integral. **[HINT: you should already know what the answer to this problem is!]**

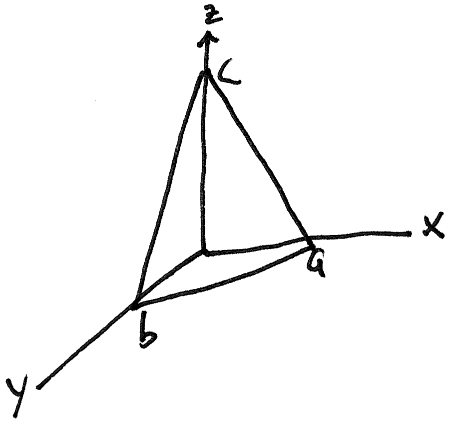
$$\int_0^b\int_0^{a(1-y/b)} 1\,dx\,dy = \int_0^a\int_0^{b(1-\frac{x}{a})} dy\,dx = \int_0^a b\left(1-\frac{x}{a}\right)dx$$

$$= -b\,\frac{1}{2}\left(1-\frac{x}{a}\right)^2\cdot(a)\Big|_0^a$$

$$= -\frac{ab}{2}\left[0^2-1^2\right] = \frac{1}{2}ab$$



(d) (*10 points.*) The equation of the plane which intersects the x-axis at a, the y-axis at b and the z-axis at c is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Show that the volume of the region bounded by this plane, the triangular region described in (b) and the $x=0$ and $y=0$ planes is equal to $\frac{1}{6}abc$.



$$\int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx = \int_0^a c\left(\left(1-\frac{x}{a}\right)y-\frac{y^2}{b}\right)\Big|_0^{b(1-\frac{x}{a})}dx$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right)^2-\frac{1}{b}\left[b\left(1-\frac{x}{a}\right)\right]^2\frac{1}{2}\,dx = c\int_0^a b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2 dx$$

$$= c\int_0^a \frac{b}{2}\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left(1-\frac{x}{a}\right)^3\cdot\frac{1}{3}(-a)\Big|_0^a$$

$$= -\frac{1}{6}abc\left[0^3-1^3\right] = \frac{1}{6}abc$$

3. Div, Grad, Curl, Green's Theorem, Fundamental Theorem of Line Integrals.
30 points.
SKILS: ANALYSIS/CRITICAL THINKING, COMPUTATION.]

(a) (*6 points.*) By direct differentiation, compute **curl** $\vec{F} = \vec{\nabla} \times \vec{\nabla} f$ for $\vec{F}(x, y) = (F_1(x, y), F_2(x, y), 0)$.

$F_1 = f_x$
$F_2 = f_y$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & 0 \end{vmatrix} = \hat{i}\left(0 - \frac{\partial}{\partial z} f_y\right) - \hat{j}\left(0 - \frac{\partial}{\partial z} f_x\right) + \hat{k}\left(\frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x\right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

(b) (*8 points.*) Consider a line integral of the gradient field $\vec{F} = \vec{\nabla} f$ on the boundary ∂R of an enclosed region R, i.e. $\oint_{\partial R} \vec{F} \cdot d\vec{x}$. What is the value of this line integral?

$$\int_{\partial R} \vec{\nabla} f \cdot d\vec{x} = 0$$ by the Fundamental Theorem of line lintegrals

(c) (*8 points.*) Use Green's Theorem to convert your line integral in part (b) to a double integral (over a particular area). Write down this integral and evaluate it. What is the value of this double integral?

$$\iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dA \overset{\text{Green's Theorem}}{=} \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_{\partial R} \vec{F} \cdot d\vec{x} \overset{\text{FTLI}}{=} \int_{\partial R} \vec{\nabla} F \cdot d\vec{x} = 0$$

$$= \iint 0\, dA$$
$$= 0$$

(d) (*8 points.*) What do your answers in (a), (b) and (c) tell you about the value of **curl grad** f?

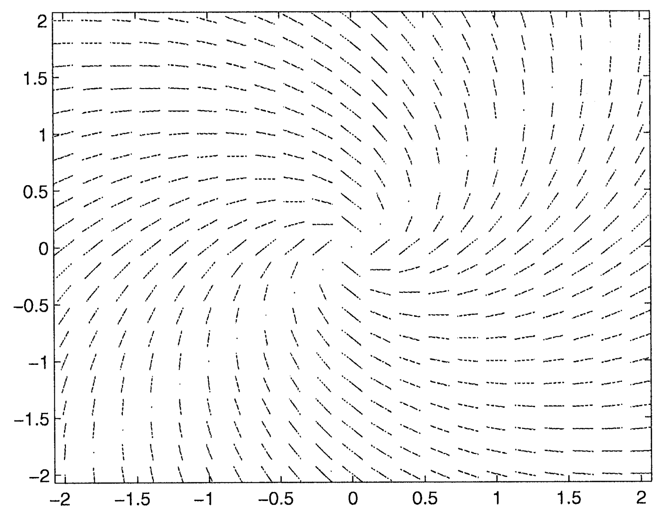
Curl of a gradient must equal zero

$$\int_{\partial R} \nabla f \cdot d\vec{x} = \iint_R (\vec{\nabla} \times \vec{\nabla} f) \cdot \hat{k}\, dA = 0$$

Thus $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$

BONUS QUESTION. Vector Fields and Vector Calculus. (*10 points.*)
SKILLS: VISUALIZATION, ANALYSIS, COMPUTATION.
Answer one of the questions below.

(I) Consider the vector field below. Discuss what properties (divergence, gradient, curl) of the field can be determined from the picture.



This field is a gradient field. The curl is zero (gradient fields. The divergence is not zero.

Particles do not repeat positions and "flow" in a direction away/towards origin.

OR

(II) Show that **div curl** $\vec{F} = 0$ always. In other words, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ for every vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

$$\nabla\cdot(\nabla\times\vec{F}) = \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= F_{3_{xy}} - F_{2_{zx}} + \left[F_{3_{xy}} - F_{1_{zy}}\right) + F_{2_{xz}} - F_{1_{yz}}$$

$$= F_{3_{xy}} - F_{3_{xy}} - F_{2_{zx}} + F_{2_{zx}} + F_{1_{zy}} - F_{1_{yz}}$$

$$= 0$$