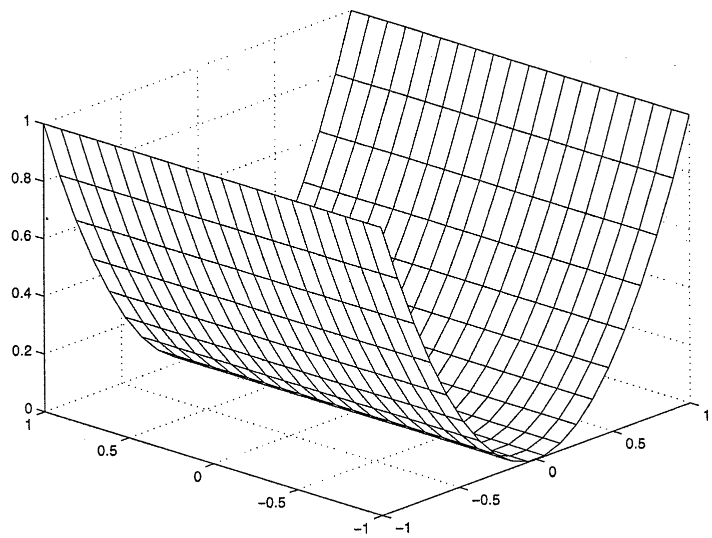
Test 2: Multivariable Calculus

Math 212 — Friday November 18 2005
Ron Buckmire — 9:30pm-10:30am

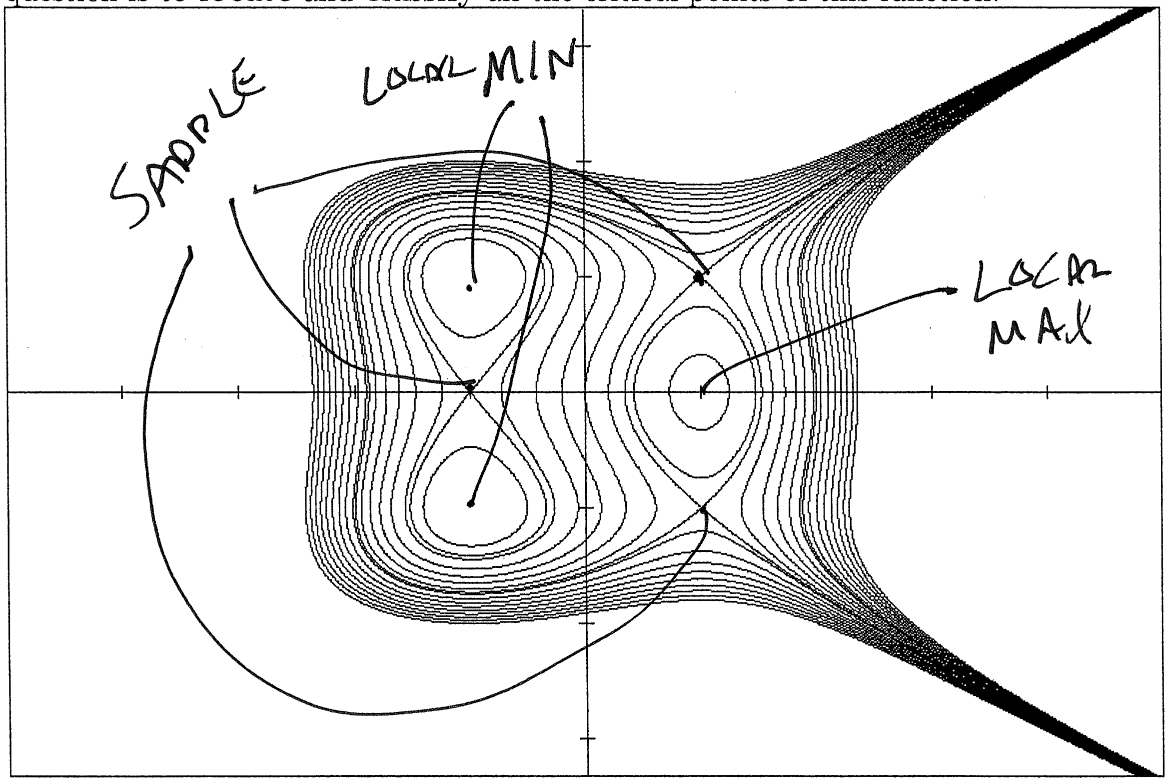
Name: Key

Directions: Read **all** problems first before answering any of them. The easiest question for you may not be at the beginning. Note how many points each problem is worth. You are enganged in an optimization problem! There are 7 pages in this test. This is a one hour, open-notes, open book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."



No.	Score	Maximum
1		20
2		20
3		20
4		20
5		20
BONUS		10
Total		**100**

1. (*20 points.*) **Unconstrained Multivariable Optimization.**
Consider $f(x, y) = 3x - x^3 - 2y^2 + y^4$ and its contour plot given below. The goal of this question is to **locate** and **classify** all the critical points of this function.



a. (*20 points*) Indicate the location of the critical points of $f(x, y)$ on the above diagram and classify each critical point as a saddle, local maximum or local minimum. **HINT: think about how you would do this problem if you did NOT have the contour diagram.**

$f_x = 3 - 3x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$f_y = -4y + 4y^3 = 0 \quad (y^2 - 1)y = 0 \Rightarrow y = 0, \pm 1$

six critical points: at
(1, 0) MAX
(1, 1) SADDLE
(1, −1) SADDLE
(−1, 0) SADDLE
(−1, 1) MIN
(−1, −1) MIN

contours cross

$f_{xx} = -6x$

$f_{yy} = -4 + 12y^2$

$f_{xy} = f_{yx} = 0$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$
$= (-6x)(12y^2 - 4)$

$D(1,0) = -6 \cdot (-4) = 24 > 0$ MAX
$f_{xx}(1,0) = -6 < 0$

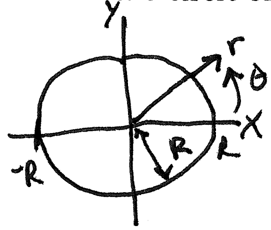
$D(-1,1) = +6 \cdot (8) > 0$
$f_{xx}(-1,1) > 0 \Rightarrow$ MIN

$D(-1,-1) = (6)(16) > 0$
$f_{xx}(-1,-1) = 6 > 0 \Rightarrow$ MIN

2. (*20 points.*) **Multiple Integration.**

The goal of this question is to evaluate $\int_0^\infty e^{-x^2}\,dx = \lim_{a\to\infty}\int_0^a e^{-x^2}\,dx$.

(a) (*10 points.*) Find $I(R) = \iint_{D_R} e^{-(x^2+y^2)}\,dx\,dy$ when D_R is $x^2+y^2 \le R^2$ (the interior of the circle of radius R centered at the origin). **HINT: pick a useful coordinate system!**



$$\int_0^{2\pi}\int_0^R e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \frac{e^{-r^2}}{-2}\Big|_0^R\,d\theta = \int_0^{2\pi} \frac{e^{-R^2}-1}{-2}\,d\theta$$

$dxdy \to r\,dr\,d\theta$

$$= 2\pi\cdot\frac{e^{-R^2}-1}{-2}$$

$$= \pi\left(1-e^{-R^2}\right)$$

(b) (*5 points.*) Take your answer $I(R)$ to **(b)** and then let $R\to\infty$. What is $\lim_{R\to\infty}\iint_{D_R} e^{-(x^2+y^2)}dx\;dy$?

$$\lim_{R\to\infty} I(R) = \pi\left(1-e^{-R^2}\right) = \pi$$

$\left(\text{Since } \lim_{R\to\infty} e^{-R^2} = 0\right)$

(c) (*5 points.*) Given that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2$ then what is the value of $\int_0^\infty e^{-x^2}\,dx$?

$$\pi = \left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 \Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

since e^{-x^2} is symmetric about $x=0$

3. (*20 points.*) **Chain Rule, Implicit Function Theorem.**
Consider a surface implicitly defined as $F(x, y, z) = 0$ which can be written as $z = f(x, y)$ and $y = g(x, z)$ and $x = h(y, z)$. If F_x, F_y, F_z exist and are not zero, the goal of this question is to obtain the (very cool) result

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1.$$

a. (*9 points*) By considering $F(x, y, z) = 0$ as $F(x, g(x, z), z) = 0$ (i.e. there is no y dependence in F) use the Chain Rule (or the implicit function theorem) to show that

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

$F(x, g(x,z), z) = 0$

$\frac{dF}{dx} = F_x + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = \frac{d0}{dx} = 0$

But $F_y = 0$ since we are defining $y = g(x,z)$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

Implicit Function Theorem
$F(x, z) = 0 \quad \vec{x} = x \quad \vec{y} = z$
$z = G(x)$
$G'(x) = \frac{\partial z}{\partial x} = -(F_{\vec{y}})^{-1} F_{\vec{x}}$
$= -(F_z)^{-1} F_x = -\frac{F_x}{F_z}$

b. (*6 points*) Through similar reasoning, simply write down similar expressions for $\frac{\partial x}{\partial y}$ and $\frac{\partial y}{\partial z}$ in terms of F_x, F_y and F_z. **HINT:** $\frac{\partial x}{\partial y} = \frac{1}{\frac{\partial y}{\partial x}}$.

$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$

$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$

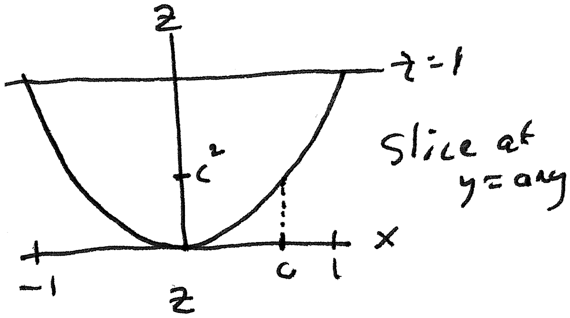
(c.) (*5 points*) Use your answers in (a) and (b) to verify the result $\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$ is true.

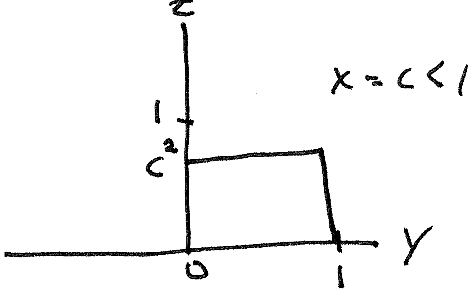
$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -\frac{\cancel{F_x}}{\cancel{F_z}} \cdot -\frac{\cancel{F_y}}{\cancel{F_x}} \cdot -\frac{\cancel{F_z}}{\cancel{F_y}} = -1$ (neat, huh?)

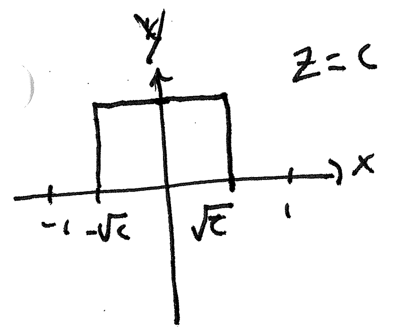
4. (*20 points.*) **Iterated Integration.**

Consider the iterated integral for $V = \int_{-1}^{1}\int_{0}^{1}\int_{x^2}^{1} dz\ dy\ dx = \frac{4}{3}$

(a) (*12 points.*) Write down 3 (**three**) of the 5 (**five**) other possible triple iterated integrals which represent the exact same value V. **HINT: There is no dependence of z upon y)** DO NOT EVALUATE THESE INTEGRALS.







1st simplest, switch y & dx!

$$\int_0^1\int_{-1}^1\int_{x^2}^1 dz\,dx\,dy$$

$$= \int_0^1\int_0^1\int_{-\sqrt{z}}^{\sqrt{z}} dx\,dz\,dy$$

$$= \int_0^1\int_0^1\int_{-\sqrt{z}}^{\sqrt{z}} dx\,dy\,dz$$

$$= \int_{-1}^1\int_{x^2}^1\int_0^1 \cancel{dz\,dx\,dy}\ dy\,dz\,dx$$

$$= \int_0^1\int_{-\sqrt{z}}^{\sqrt{z}}\int_0^1 dy\,dx\,dz$$

(b) (*8 points.*) Use any one of the iterated integrals you wrote down in part (a) to confirm the value of V.

$$\int_0^1\int_{-1}^1 1 - x^2\,dx\,dy = \int_0^1 x - \frac{x^3}{3}\Big|_{-1}^{1} dy = \int_0^1\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)dy$$

$$= \int_0^1 \frac{4}{3}\,dy = \frac{4}{3}$$

$$\int_0^1\int_0^1\int_{-\sqrt{z}}^{\sqrt{z}} dx\,dz\,dy = \int_0^1\int_0^1 2\sqrt{z}\,dz\,dy = \int_0^1 2\cdot\frac{2}{3}z^{3/2}\Big|_0^1 dy$$

$$= \int_0^1 \frac{4}{3}\,dy = \frac{4}{3}$$

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
Recall the Cobb-Douglas function $P(L,K) = bL^{\alpha}K^{1-\alpha}$ where the total production P of a certain product depends on the amount of labor L used and the amount K of capital investment ($0 < \alpha < 1$ and $b > 0$.)
If the cost of a unit of labor is m and the cost of unit of capital is n, given that the production of the company is fixed at a level Q, what values of L and K will minimize the cost function $C(L,K) = mL + nK$?

a. (*10 points*) Write down the equations you need to solve simultaneously to find the answer to the question.

Lagrange Multipliers

objective: $C(L,K) = mL + nK$

constraint: $Q = bL^{\alpha}K^{1-\alpha} \Rightarrow g(L,K) = bL^{\alpha}K^{1-\alpha} - Q$

$\vec{\nabla} C = \lambda \vec{\nabla} g$

$C_L = \lambda g_L \Rightarrow m = \lambda(\alpha b L^{\alpha-1}K^{1-\alpha})$

$C_K = \lambda g_K \quad n = \lambda(bL^{\alpha}(1-\alpha)K^{-\alpha})$

$Q = bL^{\alpha}K^{1-\alpha}$

b. (*10 points*) Solve the equations to find the values of L and K which minimize the cost function $C(L,K)$. (**HINT: Eliminate the Lagrange Multiplier first**).

$m = \lambda \alpha b \left(\frac{L}{K}\right)^{\alpha-1} \Rightarrow \frac{m}{\alpha b}\left(\frac{L}{K}\right)^{1-\alpha} = \lambda = \frac{n}{b(1-\alpha)}\left(\frac{L}{K}\right)^{-\alpha}$

$n = \lambda b(1-\alpha)\left(\frac{L}{K}\right)^{\alpha}$

$\frac{m}{\alpha}\frac{L}{K} = \frac{n}{(1-\alpha)}$

$mL = Kn\frac{\alpha}{1-\alpha}$

$\frac{L}{K} = \frac{n\alpha}{m(1-\alpha)}$

$Q = b\left(\frac{L}{K}\right)^{\alpha}K$

$Q = b\left(\frac{n\alpha}{m(1-\alpha)}\right)^{\alpha}K$

$\boxed{\frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{-\alpha} = K} \Rightarrow L = \frac{n\alpha}{m(1-\alpha)} \cdot K = \frac{n\alpha}{m(1-\alpha)}\frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{-\alpha}$

$\boxed{L = \frac{Q}{b}\left(\frac{n\alpha}{m(1-\alpha)}\right)^{1-\alpha}}$

EXTRA CREDIT (*10 points.*) **Vector Fields, Gradient Fields**

a. (*10 points*) Is the function $\vec{F}(x,y) = \begin{bmatrix} y^2 - 2xy \\ 3xy - 6x^2 \end{bmatrix}$ a gradient field? **Prove and Explain Your Answer.**

If $\vec{F}$ is a gradient field $\vec{F} = \vec{\nabla}\phi$

$\phi_x = y^2 - 2xy$ and $\phi_y = 3xy - 6x^2$

$\Rightarrow \phi = xy^2 - x^2y + f(y) \qquad \phi = \frac{3xy^2}{2} - 6x^2y + g(x)$

NOT EQUAL

NOT EQUAL

ϕ does not exist, therefore $\vec{F}$ is not a gradient field.