

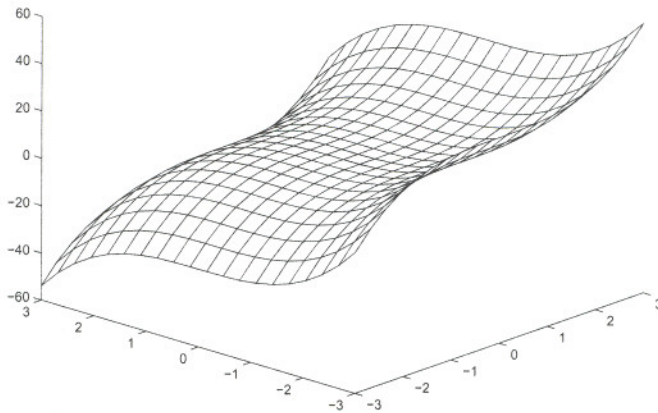
# Test 1: Multivariable Calculus

Math 212  
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Friday October 14 2005  
9:30pm-10:30am

Name: \_\_\_\_\_

**Directions:** Read *all* problems first before answering any of them. Questions 2-4 are all related, but different. There are 7 pages in this test. This is a one hour, open-notes, open book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and **CLEARLY** indicate your final answers to be graded from your “scratch work.”



No.	Score	Maximum
1		30
2		20
3		20
4		30
BONUS		5
<b>Total</b>		<b>100</b>

1. Equation of Planes, Vector Operations. You are given the following three points in the plane:

$$A = (1, 2, 3) \quad B = (2, 2, 5) \quad C = (-1, 3, 4)$$

(a) (6 points.) Find the vector  $\vec{v}$  which starts at A and points to B, and the vector  $\vec{w}$  which starts at A and points to C.

$$\vec{v} = B - A = (2, 2, 5) - (1, 2, 3) = (1, 0, 2)$$

$$\vec{v} = \hat{i} + 2\hat{k}$$

$$\vec{w} = C - A = (-1, 3, 4) - (1, 2, 3) = (-2, 1, 1)$$

$$\vec{w} = -2\hat{i} + \hat{j} + \hat{k}$$

(b) (4 points.) Find  $\vec{v} \cdot \vec{w}$ . Explain in complete sentences what this tells you about the angle between the two vectors and why.

$$\begin{aligned}\vec{v} \cdot \vec{w} &= 1 \cdot (-2) + 0 \cdot 1 + 2 \cdot 1 \\ &= -2 + 0 + 2 \\ &= 0\end{aligned}$$

$\vec{v}$  and  $\vec{w}$  are orthogonal, i.e. the angle between them is  $90^\circ$ .

(c) (10 points.) Find  $\vec{v} \times \vec{w}$ . Explain in complete sentences what this tells you about the area of the triangle ABC.

$$\begin{aligned}\vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -2 & 1 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\ &= \hat{i}(-2) - \hat{j}(1 - (-2)) + \hat{k}1 \\ &= -2\hat{i} - 5\hat{j} + \hat{k} \\ &= (-2, -5, 1)\end{aligned}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{v} \times \vec{w}| = \frac{1}{2} \sqrt{4 + 25 + 1} = \frac{1}{2} \sqrt{30}$$

(d) (10 points.) Find the equation of the plane that contains all three points A, B, and C.

We know  $(-2, -5, 1)$  is normal to the plane containing

$$\begin{aligned}-2(x-1) - 5(y-2) + 1(z-3) &= 0 \\ -2x + 2 - 5y + 10 + z - 3 &= 0 \\ -2x - 5y + z + 9 &= 0\end{aligned}$$

2. Level Sets, Vertical Slices. For the rest of the exam we will be considering the surface  $z = f(x, y) = x^3 - y^3$ .

(a) (10 points.) Identify which of the following graphs represents the level sets  $f(x, y) = k$  or different vertical slices ( $x = k$  or  $y = k$ ) of the function for  $k = -2, -1, 0, 1, 2$ .

Figure 1

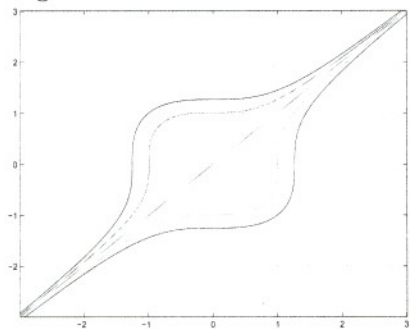


Figure 2

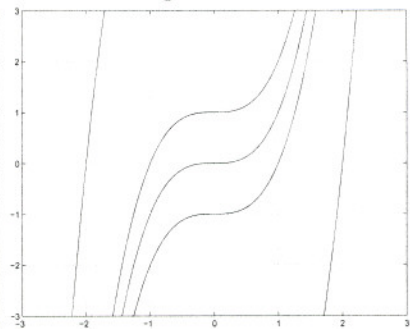
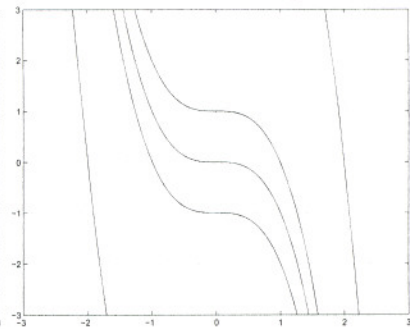


Figure 3



CLEARLY LABEL WHICH GRAPH REPRESENTS HOLDING WHICH VARIABLE CONSTANT AND FULLY EXPLAIN YOUR CHOICE BELOW.

Figure 1 is level sets  $z = k$   
 Figure 2 is vertical slices ( $y = k$ )  
 Figure 3 is vertical slices ( $x = k$ )  
 The difference between Fig 2 and Fig 3 is Fig 2 looks like  $z = x^3 - k$  while Fig 3 looks like  $z = k - y^3$

(b) (10 points.) Explain how you can use the figures above to estimate that  $f_{xy} = f_{yx} = 0$  at the origin  $(0, 0)$ . What is another way you could show this result is true everywhere in the  $(x, y)$ -plane?

$$f = x^3 - y^3$$

$$f_x = 3x^2 \quad f_{xy} = 0$$

$$f_y = -3y^2 \quad f_{yx} = 0$$

Fig 2: At the origin the rate of change of  $z$  with  $x$  near  $x = 0$  is the same as one moves vertically, so  $\frac{\Delta}{\Delta y} \left( \frac{\Delta z}{\Delta x} \right) = 0$ . A similar argument applies to Fig. 1:  
 $\frac{\Delta}{\Delta x} \left( \frac{\Delta z}{\Delta y} \right) = 0$ .

### 3. Tangent Plane Approximation.

(a) (10 points.) Find the equation of the tangent plane to the surface  $f(x, y) = x^3 - y^3$  at the point  $(x, y) = (1, 2)$ .

$$f(1, 2) = 1^3 - 2^3 = -7 \quad f_x = 3x^2 \quad f_y = -3y^2$$
$$f(x, y) \approx T(x, y) = f(1, 2) + f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$$
$$T(x, y) = -7 + 3(x-1) - 12(y-2)$$

(b) (10 points.) Use this tangent plane approximation of the surface at this point to **estimate** the value of  $(0.9)^3 - (1.99)^3$ . [Note: I know the exact value is -7.151599. I'm looking for an estimate of this using the tangent plane approximation.]

$$\begin{aligned} T(0.9, 1.99) &= -7 + 3(0.9-1) - 12(1.99-2) \\ &= -7 + 3(-0.1) - 12(-0.01) \\ &= -7 - 0.3 + 0.12 \\ &= -7.3 + 0.12 \\ &= -7.18 \end{aligned}$$

#### 4. Gradient, Directional Derivative.

(a) (15 points.) The gradient of a function  $f(x, y)$  evaluated at a point  $(x_0, y_0)$  is a vector pointing in the direction of the maximal rate of change of this function  $f(x, y)$  at the point  $(x_0, y_0)$ .

In what direction would you go from the point  $(1, 2)$  to follow the maximal rate of change on  $f(x, y) = x^3 - y^3$ ? What is the magnitude of this maximal rate of change?

$$\nabla f(x, y) = (3x^2, -3y^3)$$

$$\nabla f(1, 2) = (3, -12)$$

If you go in the direction  $3\hat{i} - 12\hat{j}$  that is the direction of maximal rate of change of  $f$ .

$$|\nabla f(1, 2)| = \sqrt{3^2 + 12^2} = \sqrt{9 + 144} = \sqrt{153}$$

(b) (10 points.) What would the rate of change have been if you went in the direction  $\vec{w} = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}$ ?

$$\begin{aligned} \frac{df}{d\vec{w}} &= \nabla f \cdot \vec{w} = (3, -12) \cdot \left(\frac{4}{5}, -\frac{3}{5}\right) \\ &= \frac{12}{5} + \frac{36}{5} = \frac{48}{5} = \boxed{9.6} \end{aligned}$$

(c) (5 points.) What is a vector direction you can move in if you want the rate of change of  $f(x, y) = x^3 - y^3$  at  $(1, 2)$  to be zero?

The direction to move in would be

such that  $\nabla f(1, 2) \cdot \vec{v} = 0$  where  $\vec{v} = (v_1, v_2)$

$$3v_1 - 12v_2 = 0 \quad \vec{v} = (12, 3)$$

**BONUS QUESTION. Continuity, Set Theory.** (5 points.)

Consider  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  where  $g(x, y) = \frac{f(x, y)}{x - y} = \frac{x^3 - y^3}{x - y}$ . Describe the domain of the function  $g(x, y)$ . What kind of set (open, closed, et cetera) is it? Is the function  $g(x, y)$  continuous on this domain? **EXPLAIN YOUR ANSWER THOROUGHLY, EXTRA CREDIT POINTS ARE HARD TO GET.**

The domain is  $(x, y) \in \mathbb{R}^2 \setminus \{y = x\}$ . All  $(x, y)$  pairs in  $\mathbb{R}^2$  except where  $x = y$ .

This set is open. Every point in the set has a neighborhood which only contains points of the set.

The set is NOT CLOSED.

The points along  $y = x$  are all limit points of the set but are NOT in the set, thus the set does not

$g(x, y)$  IS continuous everywhere in its

domain.

$$g(x, y) = \frac{x^3 - y^3}{x - y} = \frac{(x - y)(x^2 + xy + y^2)}{x - y}$$

$$g(x, y) = x^2 + xy + y^2$$